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A mixed Hammerstein integral equation

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Abstract. A sufficient condition for the existence and uniqueness of a continuous solution of the integral equation

$$f(x) = G(x,\ h_1(x) + \int_D K_1(x,y)H_1(y, f(y))dy,\ h_2(x) + \int_{D\cap(-\infty,x]} K_2(x,y)H_2(y, f(y))dy)$$

is established under regularity conditions on the functions G, h_1, h_2, H_1, H_2, and on the kernels K_1 and K_2 where D is a subset of $\mathbb{R}^n$ and $(-\infty, x]$, $x \in \mathbb{R}^n$, is a simplified notation for the interval $\prod_{i=1}^n(-\infty, x_i] \subset \mathbb{R}^n$.
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1. Introduction

The non linear integral equation $f(x) = \int_a^b K(x,y)H(y, f(y))dy$ has been studied by R.Iglish [1], A.Hammerstein, [2], M.Golomb, [3] and C.L.Dolph [4]. Under restrictive conditions on the kernel and controlling the non linearity of the function H, they succeeded in finding sufficient conditions either to existence and uniqueness of a solution or solely to the existence of solutions to this integral equation. Non linearities of the type $H(y, f(y)) = 1/y$ lead to singular integral equations. The integral equation $f(x)\int_0^1 f(y)K(x,y)dy = 1$ that arises in the theory of communication systems was studied in [5] by P. Nowosad where the existence and uniqueness of continuous positive real solutions was established for positive semidefinite symmetric non-negative kernels, $K(x,y)$, $0 \le x, y \le 1$, such that

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$\int_0^1 K(x,y)dy \geq \delta > 0$. An extension of this result was obtained by S. Karlin and L. Nirenberg in [6]. In their work they prove the existence of continuous positive solutions of the equation $f(x)\int_0^1 f(y)^\alpha K(x,y)dy = 1$ where α is a fixed positive parameter and $K(x,y)$ is a non-negative continuous function on $[0,1]^2$ such that $K(x,x) > 0$ for all $x \in [0,1]$. They also showed the uniqueness of continuous positive solutions in case the parameter α belongs to $(0,1]$. The Schauder fixed point theorem was used to derive the existence of solutions. Further extensions of P. Nowosad 's result can be found in [7] where positive solutions are established for the integral equation $f(x) = g(x) + \int_0^1 K(x,y)\left(\frac{1}{(f(y))^\alpha} + h(f(y))\right)dy, \alpha > 0, x \in [0,1]$. An existence theorem of integrable solutions to the integral equation $f(x) = g(x) + \lambda \int_D K(x,y)H(y,f(y))dy$ where $D \subset \mathbb{R}^n$ is a compact set and g, K, and H are functions with values in finite dimensional Banach spaces is obtained by G.Emmanuele in [8]. Conditions for the existence of nonzero solutions of integral equations of the the form $f(x) = \int_D K(x,y)H(y,f(y))dy$, D compact subset of $\mathbb{R}^n$, where K is a real valued function that changes sign and may be discontinuous, and H satisfies Caratheodory conditions, are presented by G.Infante and J.R.L.Webb in [9]. The existence of integrable solutions to the non linear integral equation of Hammerstein - Volterra type $f(x,t) = \int_0^1 K(x,y)H(y,f(y,t))dy + \int_0^t F(t,z)f(x,z)dz$ is obtained by M.A.Abdou, W.G.El-Sayed, and E.I.Deebs in [10]. Also of interest are monotone solutions to integral equations. In [11], J.Banas , J. Caballero, J.Rocha, and K. Sadaragani established the existence of nondecreasing continuous solutions on a bounded and closed interval I to the nonlinear integral equation of Volterra type $f(x) = a(x) + (Tf)(x)\int_0^x v(x,y,f(y))dy$, $y \in I$, under a set of conditions on the functions a, v, and on the continuous operator $T : C(I) \to C(I)$. A similar result is presented by W.G.El-Sayed and B.Rzepka in [12] for the quadratic integral equation of Urysohn type with the form $f(x) = a(x) + H(x,f(x))\int_0^1 u(x,y,f(y))dy,\ y \in I$. Due to plenty of practical applications, numerical methods for solving integral equations are of great interest. Recently, S.Yousefi and M.Razzaghi, [13] and K.Maleknejad and H.Derili, [14] applied wavelet methods to obtain numerical solutions to Volterra - Fredholm and Hammerstein type integral equations. In this short article we study the integral equation:

$$\begin{aligned} f(x) = G(x,\ h_1(x) + \int_D K_1(x,y)H_1(y,f(y))dy,\ h_2(x) \\ + \int_{D\cap(-\infty,x]} K_2(x,y)H_2(y,f(y))dy) \end{aligned}$$

where $f : D \subset \mathbb{R}^n \to \mathcal{A}$ is a function with values in a complete normed finite dimensional algebra $\mathcal{A}$, $K_i : D^2 \to \mathcal{A}$, $h_i : D \to \mathcal{A}$, $H_i : D \times Im(G) \to \mathcal{A}$, for $i \in \{1, 2\}$ and $G : D \times \mathcal{R}_1 \times \mathcal{R}_2 \to \mathcal{A}$, $\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{A}$ satisfy regularity conditions. Throughout this work we denote the $\mathbb{R}^n$ interval $\prod_{i=1}^n (a_i, b_i]$ by $(a, b]$ where $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in \mathbb{R}^n$. Also, $a_i \wedge b_i$ will denote $\min\{a_i, b_i\}$ and $a_i \vee b_i$ equals $\max\{a_i, b_i\}$. For vectors this is done componentwise: $a \wedge b = (a_1 \wedge b_1, ..., a_n \wedge b_n)$ and $a \vee b = (a_1 \vee b_1, ..., a_n \vee b_n)$. It is not required neither that the algebra contains a unit element nor that, in case it has one, that its norm be one. These are common requirements to call $\mathcal{A}$ a Banach algebra. Being $(\mathcal{A}, +, \cdot, \times, \| \|)$ an algebra over a field $\mathcal{K}$ endowed with an absolute value $| \, |$, we suppress the notational use of . and $\times$ and use xy and αx instead of $x \times y$ and $\alpha.x$ respectively. We assume that for all $x, y \in \mathcal{A}$, $\|xy\| \leq \|x\|\|y\|$. We use the notation $B[x, r]$ to mean the closed ball centered at x with radius r in a metric space.

In section 2 we present the main result: an existence and uniqueness of solution theorem based only on Banach's fixed point theorem. In section 3 some related results and corollaries are obtained, and, in section 4, we conclude this work with some examples and final remarks.

2. **Main results**

The following elementary lemma is used in the proof of Theorem 2.2.

Lemma 2.1. *Let* $(\mathbb{R}^n, \Lambda, \ell)$ *be* $\mathbb{R}^n$ *endowed with Lebesgue measure and* σ*-algebra,* $A \in \Lambda$*,* $f : A \to \mathbb{R}_+$ *such that* $\int_A f d\ell < \infty$ *and* $k \in \mathbb{N}$*. Then, for all Lebesgue measurable sets* $C \subset \bigcup_{i=1}^k C_i$ *where, for all* $i, 1 \leq i \leq k$*,* $C_i = H_i \times \{x_i + tn_i : 0 \leq t \leq \delta_i\}$*,* H_i *a hyperplane,* $x_i \in H_i$ *and* n_i *one of its unitary normals, we have*

$$\int_C f d\ell \to 0 \quad as \quad \max\{\delta_i : 1 \leq i \leq k\} \to 0.$$

Proof : If A is bounded this is a direct consequence of the fact that, for $C \subset A$, we have $\lim_{\ell(C) \to 0} \int_C f d\ell = 0$ whenever $\int_A f d\ell < \infty$. In case A is not bounded, observe that for all $\epsilon > 0$ there exists $r > 0$ such that $\int_{\mathbb{R}^n \setminus [-r,r]^n} f d\ell < \epsilon/2$ and , taking into account that $\ell(C \cap [-r, r]^n) < k \ \max\{\delta_i : 1 \leq i \leq k\}(2r\sqrt{n})^{n-1}$, we guarantee that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\max\{\delta_i : 1 \leq i \leq k\} < \delta$ then $\int_{C \cap [-r,r]^n} f d\ell < \epsilon/2$, and, consequently, we conclude that

$$\forall \, \epsilon > 0 \; \exists \, \delta > 0 \;\; \max\{\delta_i : 1 \leq i \leq k\} < \delta \; \longrightarrow \; \int_C f d\ell < \epsilon.$$

Our main result, an application of the fixed point theorem for contraction mappings, is the following:

Theorem 2.1. *Let D be a compact subset of $\mathbb{R}^n$, $\mathcal{A}$ be a finite dimensional complete normed algebra, $K_i : D^2 \to \mathcal{A}$, $h_i : D \to \mathcal{A}$, $H_i : D \times \mathcal{A} \to \mathcal{A}$, for $i \in \{1,2\}$, and $G : D \times \mathcal{A}^2 \to \mathcal{A}$ be functions such that:*

(1) $\forall i \in \{1,2\}\ \forall x \in D \quad \lim_{z\to x} \int_D \|K_i(z,y) - K_i(x,y)\|dy = 0.$
(2) $\forall i \in \{1,2\}\ \| \int_D \|K_i(x,y)\|dy\|_\infty < \infty.$
(3) $\forall i \in \{1,2\}\ \sup\{\|H_i(x,z)\| : x \in D, z \in Im(G)\} < \infty$
(4) $\forall i \in \{1,2\}\ \exists\ \iota_i\ \iota_i \geq 0\ \forall z \in D\ \forall x, y \in Im(G)$
$\|H_i(z,x) - H_i(z,y)\| \leq \iota_i \|x - y\|.$
(5) *G is continuous in* $D \times (Im(h_1) + B[0, \nu_1\kappa_1]) \times (Im(h_2) + B[0, \nu_2\kappa_2])$.
(6) $\forall i \in \{1,2\}\ \exists\ \mu_i \geq 0\ \forall x\ \in D\ \forall\ y_i, z_i\ \in Im(h_i) + B[0, \nu_i\kappa_i],$
$\|G(x,y_1,z_1) - G(x,y_2,z_2)\| \leq \mu_1\|y_1 - y_2\| + \mu_2\|z_1 - z_2\|.$
(7) $\forall i \in \{1,2\}, h_i$ *is continuous.*

Denote, for $i \in \{1,2\}$, $\| \int_D \|K_i(x,y)\|dy\|_\infty$ by κ_i and $\sup\{\|H_i(x,z)\| : x \in D, z \in Im(G)\}$ by ν_i. Then whenever $\mu_1\kappa_1\iota_1 + \mu_2\kappa_2\iota_2 < 1$ there exists one and only one continuous function $f : D \to \mathcal{A}$ such that

$$f(x) = G\Big(x,\ h_1(x) + \int_D K_1(x,y)H_1(y,f(y))dy,\ h_2(x) + \int_{D\cap(-\infty,x]} K_2(x,y)H_2(y,f(y))dy\Big).$$

Before the proof of this theorem is given, we observe that the functions $H_i : D \times \mathcal{A} \to \mathcal{A}$, for $i \in \{1,2\}$, and $G : D \times \mathcal{A}^2 \to \mathcal{A}$ could be replaced by $H_i : D \times Im(G) \to \mathcal{A}$, for $i \in \{1,2\}$, and $G : D \times (Im(h_1) + B[0, \nu_1\kappa_1]) \times (Im(h_2) + B[0, \nu_2\kappa_2]) \to \mathcal{A}$

Proof : The possible continuous solutions to the integral equation belong to $S := C(D; \mathcal{A})$ which is a complete metric space with distance given by the supremum norm.

Now, for all $x \in D$, we have for every f

$$\begin{aligned}
0 &\le \| \int_D K_1(x,y)H_1(y,f(y))dy\| \le \int_D \|K_1(x,y)\|\|H_1(y,f(y))\|dy \\
&\le \int_D \|K_1(x,y)\| \sup\{\|H_1(s,t)\| : s \in D, t \in Im(G)\}dy \\
&= \nu_1 \int_D \|K_1(x,y)\|dy \le \nu_1 \sup\{\int_D \|K_1(x,y)\|dy : x \in D\} \\
&= \nu_1 \| \int_D \|K_1(x,y)\|dy\|_\infty = \nu_1\kappa_1.
\end{aligned}$$

Analogously,

$$0 \le \| \int_{D\cap(-\infty,x]} K_2(x,y)H_2(y,f(y))dy\| \le \nu_2\kappa_2.$$

Thus, for all $x \in D$,

$$h_1(x) + \int_D K_1(x,y)H_1(y,f(y))dy \in (Im(h_1) + B[0,\nu_1\kappa_1])$$

and

$$h_2(x) + \int_{D\cap(-\infty,x]} K_2(x,y)H_2(y,f(y))dy \in (Im(h_2) + B[0,\nu_2\kappa_2]) .$$

Let, for all $f \in S$, $T(f) : D \to \mathcal{A}$ be given by

$$\begin{aligned}
Tf(x) &:= (T(f))(x) \\
&= G(x,\ h_1(x) + \int_D K_1(x,y)H_1(y,f(y))dy,\ h_2(x) \\
&\quad + \int_{D\cap(-\infty,x]} K_2(x,y)H_2(y,f(y))dy).
\end{aligned}$$

Note that $T(S) \subset S$ for if $f \in S$ we have

$$\begin{aligned}
&\|Tf(x) - Tf(z)\| = \\
&\qquad = \|G(x,\ h_1(x) + \int_D K_1(x,y)H_1(y,f(y))dy,\ h_2(x) \\
&\qquad + \int_{D\cap(-\infty,x]} K_2(x,y)H_2(y,f(y))dy) - G(z,\ h_1(z) \\
&\qquad + \int_D K_1(z,y)H_1(y,f(y))dy,\ h_2(z) \\
&\qquad + \int_{D\cap(-\infty,z]} K_2(z,y)H_2(y,f(y))dy)\|
\end{aligned}$$

Now, the continuity of G in $D\times(Im(h_1)+B[0,\nu_1\kappa_1])\times(Im(h_2)+B[0,\nu_2\kappa_2])$ and that of h_1 and h_2 in D together with the inequalities and limits bellow

$$\begin{aligned}
&\|\left(\int_D K_1(x,y)H_1(y,f(y))dy\right)-\left(\int_D K_1(z,y)H_1(y,f(y))dy\right)\|\\
&\leq \int_D \|H_1(y,f(y))\|\|K(z,y)-K(x,y)\|dy\\
&\leq \sup\{\|H_1(y,f(y))\|:y\in D\}\int_D \|K_1(z,y)-K_1(x,y)\|dy\\
&\leq \sup\{\|H_1(s,t)\|:s\in D,t\in Im(G)\}\int_D \|K_1(z,y)-K_1(x,y)\|dy\\
&=\nu_1\int_D \|K_1(z,y)-K_1(x,y)\|dy\rightarrow 0 \text{ as } z\rightarrow x
\end{aligned}$$

and

$$\begin{aligned}
&\|(\int_{D\cap(-\infty,x]} K_2(x,y)H_2(y,f(y))dy)\\
&\qquad -(\int_{D\cap(-\infty,z]} K_2(z,y)H_2(y,f(y))dy)\|\\
&\leq \int_{D\cap(-\infty,x\wedge z]} \|H_2(y,f(y))\|\|K_2(z,y)-K_2(x,y)\|dy\\
&\qquad +\int_{D\cap(-\infty,x]\setminus D\cap(-\infty,x\wedge z]} \|H_2(y,f(y))\|\|K_2(x,y)\|dy\\
&\qquad +\int_{D\cap(-\infty,z]\setminus D\cap(-\infty,x\wedge z]} \|H_2(y,f(y))\|\|K_2(z,y)\|dy\\
&\leq \nu_2(\int_D \|K_2(z,y)-K_2(x,y)\|dy\\
&\qquad +\int_{D\cap(-\infty,x]\setminus D\cap(-\infty,x\wedge z]} \|K_2(x,y)\|dy\\
&\qquad +\int_{D\cap(-\infty,z]\setminus D\cap(-\infty,x\wedge z]} \|K_2(z,y)\|dy)\rightarrow 0
\end{aligned}$$

as $z\rightarrow x$ guarantee that Tf is continuous.

Observe that the terms

$$\int_{D\cap(-\infty,x]\setminus D\cap(-\infty,x\wedge z]} \|K_2(x,y)\|dy$$

and

$$\int_{D\cap(-\infty,z]\setminus D\cap(-\infty,x\wedge z]} \|K_2(z,y)\|dy$$

go to zero as $z \to x$ as a consequence of assumption 2 and the fact that the Lebesgue measure of the sets $D\cap(-\infty,x]\setminus D\cap(-\infty,x\wedge z]$ and $D\cap(-\infty,z]\setminus D\cap(-\infty,x\wedge z]$ are bounded above by $n\|x-z\|\,(diam(D))^{n-1}$, where $diam(D)<\infty$ stands for the diameter of the compact set D .

Now let us show that T is a contraction in S.

We have, for all $f,g\in S$

$$\begin{aligned}
&\|Tf-Tg\|_\infty = \|G(x,\ h_1(x)+\int_D K_1(x,y)H_1(y,f(y))dy,\ h_2(x)\\
&+\int_{D\cap(-\infty,x]} K_2(x,y)H_2(y,f(y))dy) - G(x,\ h_1(x)\\
&+\int_D K_1(x,y)H_1(y,g(y))dy,\ h_2(x)+\int_{D\cap(-\infty,x]} K_2(x,y)H_2(y,g(y))dy)\|_\infty\\
&\le \mu_1\|h_1(x)-h_1(x)+\int_D K_1(x,y)H_1(y,f(y))dy\\
&-\int_D K_1(x,y)H_1(y,g(y))dy\|_\infty + \mu_2\|h_2(x)-h_2(x)\\
&+\int_{D\cap(-\infty,x]} K_2(x,y)H_2(y,f(y))dy - \int_{D\cap(-\infty,x]} K_2(x,y)H_2(y,g(y))dy\|_\infty\\
&\le \mu_1\sup\{\|H_1(y,f(y))-H_1(y,g(y))\| : y\in D\}\|\int_D \|K_1(x,y)\|dy\|_\infty\\
&+\mu_2\sup\{\|H_2(y,f(y))-H_2(y,g(y))\| : y\in D\cap(-\infty,x]\}\\
&\qquad\cdot\|\int_{D\cap(-\infty,x]} \|K_2(x,y)\|dy\|_\infty\\
&\le \mu_1\sup\{\iota_1\|f(y)-g(y)\| : y\in D\}\|\int_D \|K_1(x,y)\|dy\|_\infty\\
&+\mu_2\sup\{\iota_2\|f(y)-g(y)\| : y\in D\cap(-\infty,x]\}\\
&\qquad\cdot\|\int_{D\cap(-\infty,x]} \|K_2(x,y)\|dy\|_\infty\\
&\le \mu_1\iota_1\|f-g\|_\infty\|\int_D \|K_1(x,y)\|dy\|_\infty + \mu_2\iota_2\|f-g\|_\infty\\
&\qquad\cdot\|\int_D \|K_2(x,y)\|dy\|_\infty
\end{aligned}$$

Thus
$$\|Tf - Tg\|_\infty \leq (\mu_1\iota_1\kappa_1 + \mu_2\iota_2\kappa_2)\,\|f-g\|_\infty,$$
a contraction whenever $\mu_1\iota_1\kappa_1 + \mu_2\iota_2\kappa_2 < 1$, and the theorem follows.

Theorem 2.2. *Let D be a measurable subset of $\mathbb{R}^n$, $\mathcal{A}$ be a finite dimensional complete normed algebra, $K_i : D^2 \to \mathcal{A}$, $h_i : D \to \mathcal{A}$, $H_i : D \times \mathcal{A} \to \mathcal{A}$, for $i \in \{1,2\}$, and $G : D \times \mathcal{A}^2 \to \mathcal{A}$ be functions satisfying conditions 1 to 4, 6 and 7 in theorem 2.1. If G is continuous and bounded in $D \times (Im(h_1) + B[0,\nu_1\kappa_1]) \times (Im(h_2) + B[0,\nu_2\kappa_2])$, then whenever $\mu_1\kappa_1\iota_1 + \mu_2\kappa_2\iota_2 < 1$ there exists one and only one continuous function $f : D \to \mathcal{A}$ such that*
$$f(x) = G(x,\ h_1(x) + \int_D K_1(x,y)H_1(y,f(y))dy,\ h_2(x) + \int_{D\cap(-\infty,x]} K_2(x,y)H_2(y,f(y))dy).$$
The same remark that follows theorem 2.1 is still applicable.

Proof : Follow the steps in theorem 2.1 proof. Observe that $S := \mathcal{B}(D;\overline{Im(G)})$, the set of bounded functions from D to the closure of the image of G, is a Banach space and that $T(S) \subset S$. Use Lemma 2.1 to deal with the possibly unbounded sets $D\cap(-\infty,x] \setminus D\cap(-\infty,x\wedge z]$ and $D\cap(-\infty,z] \setminus D\cap(-\infty,x\wedge z]$.

3. Related results and corollaries

Theorem 3.1. *Let $K_i : ([0,1]^n)^2 \to \mathbb{R}_+$, $h_i : [0,1]^n \to \mathbb{R}_+$, for $i \in \{1,2\}$, be non-negative functions, β and δ be strictly positive real numbers, and $H_i : [0,1]^n \times [0,\frac{1}{\delta}] \to \mathbb{R}_+$ for $i \in \{1,2\}$ and $G : [0,1]^n \times (Im(h_1) + [0,\nu_1\kappa_1]) \times (Im(h_2) + [0,\nu_2\kappa_2]) \to \mathbb{R}_+$ be functions such that:*

(1) $\forall i \in \{1,2\}\ \forall x \in [0,1]^n \quad \lim_{z\to x} \int_{[0,1]^n} |K_i(z,y) - K_i(x,y)|dy = 0.$
(2) $\forall i \in \{1,2\}\ \| \int_{[0,1]^n} K_i(x,y)dy\|_\infty < \infty.$
(3) $\forall i \in \{1,2\}\ \sup\{H_i(x,z) : x \in [0,1]^n\ z \in [0,\frac{1}{\delta}]\} < \infty.$
(4) $\forall i \in \{1,2\}\ \exists\ \iota_i \geq 0\ \forall z \in [0,1]^n\ \forall x,y \in [\frac{1}{\beta},\frac{1}{\delta}]\ \ |H_i(z,x) - H_i(z,y)| \leq \iota_i|x-y|.$
(5) $0 < \delta = \inf\{G(x,y,z) : (x,y,z) \in [0,1]^n \times (Im(h_1) + [0,\nu_1\kappa_1]) \times (Im(h_2) + [0,\nu_2\kappa_2])\}$ *and* $\beta = \sup\{G(x,y,z) : (x,y,z) \in [0,1]^n \times (Im(h_1) + [0,\nu_1\kappa_1]) \times (Im(h_2) + [0,\nu_2\kappa_2])\} < \infty.$
(6) $\forall i \in \{1,2\}\ \exists\ \mu_i \geq 0\ \forall x\ \in [0,1]^n\ \forall\ y_i, z_i\ \in Im(h_i) + [0,\nu_i\kappa_i]$, $|\frac{G(x,y_1,z_1)-G(x,y_2,z_2)}{G(x,y_1,z_1)G(x,y_2,z_2)}| \leq \mu_1|y_1-y_2| + \mu_2|z_1-z_2|.$
(7) $\forall i \in \{1,2\}$ h_i *is continuous*

Denote, for $i \in \{1,2\}$, $\| \int_{[0,1]^n} K_i(x,y)dy\|_\infty$ *by* κ_i *and* $\sup\{H_i(x,z) : x \in [0,1]^n \ z \in [0, \frac{1}{\delta}]\}$ *by* ν_i *Then whenever* $\mu_1\kappa_1\iota_1 + \mu_2\kappa_2\iota_2 < 1$ *there exists one and only one continuous function* $f : [0,1]^n \to \mathbb{R}$*, such that*

$$f(x)\ G(x, \quad h_1(x) + \int_{[0,1]^n} K_1(x,y)H_1(y,f(y))dy, \quad h_2(x) + \int_{[0,x]} K_2(x,y)H_2(y,f(y))dy) = 1.$$

Clearly this function is strictly positive.

Proof :

Similar to that of theorem 2.1 . For all $x \in [0,1]^n$ we have for every f

$$\begin{aligned} 0 &\le \int_{[0,1]^n} K_1(x,y)H_1(y,f(y))dy \\ &\le \int_{[0,1]^n} \sup\{H_1(y,f(y)) : y \in [0,1]^n\}K_1(x,y)dy \\ &\le \nu_1 \int_{[0,1]^n} K_1(x,y)dy \le \nu_1\kappa_1. \end{aligned}$$

Analogously,

$$0 \le \int_{[0,x]} K_2(x,y)H_2(y,f(y))dy \le \nu_2\kappa_2.$$

so that for all $x \in [0,1]^n$ the solution of the integral equation satisfies

$$\frac{1}{\beta} \le f(x) = G\Big[x, h_1(x) + \int_{[0,1]^n} K_1(x,y)H_1(y,f(y))dy, h_2(x) + \int_{[0,x]} K_2(x,y)H_2(y,f(y))dy\Big]^{-1} \le \frac{1}{\delta}.$$

Thus the possible continuous solutions to the integral equation belong to $S := C([0,1]^n; [\frac{1}{\beta}, \frac{1}{\delta}])$ which is a complete metric space with distance given by the supremum norm.

Let, for all $f \in C([0,1]^n; \mathbb{R}_+^*)$, $T(f) : [0,1] \to \mathbb{R}$ be given by

$$Tf(x) := (T(f))(x) = G\Big[x, h_1(x) + \int_{[0,1]^n} K_1(x,y)H_1(y,f(y))dy, h_2(x) + \int_{[0,x]} K_2(x,y)H_2(y,f(y))dy\Big]^{-1}.$$

Note that $T(S) \subset S$ for if $f \in S$ we have

$$0 \leq \int_{[0,1]^n} K_1(x,y)H_1(y,f(y))dy \leq \nu_1\kappa_1$$

and

$$0 \leq \int_{[0,x]} K_2(x,y)H_2(y,f(y))dy \leq \nu_2\kappa_2$$

from which

$$\delta^{-1} \geq \left\| G\left[x, h_1(x) + \int_{[0,1]^n} K_1(x,y)H_1(y,f(y))dy, h_2(x) + \int_{[0,x]} K_2(x,y)H_2(y,f(y))dy\right]^{-1}\right\|_\infty \geq \beta^{-1}$$

and we get $\beta^{-1} \leq \|Tf\|_\infty \leq \delta^{-1}$.

Moreover, conditions 5 and 6 imply the continuity of $\frac{1}{G}$ on $[0,1]^n \times (Im(h_1) + [0,\nu_1\kappa_1]) \times (Im(h_2) + [0,\nu_2\kappa_2])$ and a similar argument to that in Theorem 2.1 proof shows that $|Tf(x) - Tf(z)| \to 0$ as $z \to x$; i.e. leads to the conclusion that Tf is continuous.

Observe that$|\frac{G(x,y_1,z_1)-G(x,y_2,z_2)}{G(x,y_1,z_1)G(x,y_2,z_2)}| = |\frac{1}{G(x,y_1,z_1)} - \frac{1}{G(x,y_1,z_1)}|$ so that condition 6 in this corollary is the same as condition 6 in Theorem 2.1 applied to $\frac{1}{G}$. Now follow the steps in Theorem 2.1 proof to show that T is a contraction in S.

We observe that the conditions on the kernels in theorems 2.1 and 3.1 hypothesis are implied by Kernel continuity. As a matter of fact, their continuity on the compact set D^2 or on $[0,1]^{2n}$ implies uniform continuity on D^2 or on $[0,1]^{2n}$ which, by its turn, implies condition 1; continuity on D^2 or on $[0,1]^{2n}$ also implies boundedness and integrability so that condition 2 is guarantied. Partial differentiability of H_i with respect to the second variable on $D \times Im(G)$ or on $[0,1]^n \times [\frac{1}{\beta}, \frac{1}{\delta}]$ and boundedness of this derivative on $D \times Im(G)$ or on $[0,1]^n \times [\frac{1}{\beta}, \frac{1}{\delta}]$ implies condition 4 as we can choose, by the mean value inequality, $\iota_i = \sup\{\|\partial_2 H_i(x,z)\| : (x,z) \in D \times Im(G)\}$ or $\iota_i = \sup\{\|\partial_2 H_i(x,z)\| : (x,z) \in [0,1]^n \times [\frac{1}{\beta}, \frac{1}{\delta}]\}$. Also continuous differentiability of H_i on $[0,1]^n \times [\frac{1}{\beta}, \frac{1}{\delta}]$ clearly implies 4.

Similarly, bounded partial differentiability with respect to the second and third variables of G on $D \times (Im(h_1) + B[0,\nu_1\kappa_1]) \times (Im(h_2) + B[0,\nu_2\kappa_2])$ or of $\frac{1}{G}$ on $[0,1]^n \times (Im(h_1) + [0,\nu_1\kappa_1]) \times (Im(h_2) + [0,\nu_2\kappa_2])$ or continuous differentiability of $\frac{1}{G}$ on $[0,1]^n \times (Im(h_1) + [0, \nu_1\kappa_1]) \times (Im(h_2) + [0, \nu_2\kappa_2])$ implies condition 6.

Clearly, the pure Hammerstein or Volterra - Hammerstein integral equations are particular cases of the complete mixed integral equation.

In this way one can write some corollaries, the weaker of them is:

Corollary 3.1. *Let* $K : [0,1]^2 \to \mathbb{R}$ *be a non-negative continuous function. Denote* $\| \int_0^1 K(x,y)dy\|_\infty$ *by* κ *and assume* $\kappa < \infty$. *Let also* β *and* δ *be strictly positive real numbers, and* $H : [0, \frac{1}{\delta}] \to \mathbb{R}_+$ *and* $G : [0, \nu\kappa] \to \mathbb{R}_+$ *be functions such that:*

(1) $\nu = \sup\{H(z) : z \in [0, \frac{1}{\delta}]\} < \infty$.
(2) H *is continuously differentiable on* $[\frac{1}{\beta}, \frac{1}{\delta}]$.
Denote $\iota = \sup\{|H'(z)| : z \in [\frac{1}{\beta}, \frac{1}{\delta}]\}$.
(3) $0 < \delta = \inf\{G(z) : z \in [0, \nu\kappa]\}$ *and*
$\beta = \sup\{G(z) : z \in [0, \nu\kappa]\} < \infty$.
(4) G *is continuously differentiable on* $[0, \nu\kappa]$.
Denote $\mu = \sup\{| \left(\frac{1}{G(z)}\right)' | : z \in [0, \nu\kappa]\}$.

Then whenever $\mu\kappa\iota < 1$ *there exists one and only one continuous function* $f(x)$, $x \in [0,1]$, *such that*

$$f(x)\ G\left(\int_0^1 K(x,y)H(f(y))dy\right) = 1.$$

Clearly this function is strictly positive.

Proof : Theorem 3.1 and remarks above. ∎

4. Examples and Final Remarks

The following examples will show typical uses of the theorems and corollaries developed so far.

Concerning the integral equation

$$f(x)\ \exp\left(\int_0^1 f(y)^\gamma K(x,y)dy\right) = 1$$

we can obtain the following

Example 4.1. *Let* γ *be a real positive number and* $K : [0,1]^2 \to \mathbb{R}$ *be a non-negative continuous function such that*

$$\| \int_0^1 K(x,y)dy\|_\infty = \kappa$$

where $\kappa < \frac{1}{\gamma}$ *in case* $\gamma \geq 1$ *and* $\kappa\gamma e^{(1-\gamma)\kappa} < 1$ *in case* $0 < \gamma < 1$.

Then there exists one and only one continuous solution $f(x)$, $x \in [0,1]$, *to the integral equation*

$$f(x)\ \exp\left(\int_0^1 f(y)^\gamma K(x,y)dy\right) = 1.$$

This solution is strictly positive.

Proof :

Clearly, $\delta = \inf\{\exp(z) : z \in \mathbb{R}_+\} = \inf\{\exp(z) : z \in [0,a]\} = 1$, whatever $a > 0$ is, so that $\nu = \sup\{z^\gamma : z \in [0, \frac{1}{\delta}]\} = 1$ and $\beta = \sup\{\exp(z) : z \in [0, \nu\kappa]\} = e^\kappa$. Thus $\iota = \sup\{\gamma z^{\gamma-1} : z \in [\frac{1}{e^\kappa}, 1]\}$ which is equal to γ in case $\gamma \geq 1$ and to $\gamma e^{(1-\gamma)\kappa}$ in case $0 < \gamma < 1$. The exponential function is continuously differentiable and $\mu = \sup\{e^{-x} : x \in [0,\kappa]\} = 1$. Now apply Corollary 3.1 .
The second example concerns integral equations for matrix valued functions.

Example 4.2. *Consider the complete mixed Hammerstein integral equation on* $M_{2\times2}(\mathbb{R})$*-valued functions of* $[0,1]^2$

$$\begin{aligned} f(x) = G(x, h_1(x) &+ \int_{[0,1]^2} K_1(x,y)H_1(y,f(y))dy, h_2(x) \\ &+ \int_{[0,x]} K_2(x,y)H_2(y,f(y))dy) \end{aligned}$$

where $\lambda_1, \lambda_2, \theta_1, \theta_2$ *are real numbers,* $K_1(x,y) = \exp(\lambda_1 \begin{pmatrix} x_1 & y_1 \\ y_2 & x_2 \end{pmatrix})$,

$H_1(y,f(y)) = \frac{\theta_1 f(y)^2}{1+\|f(y)\|^2} \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$, $h_1(x) = \begin{pmatrix} 2+x_1 & \frac{\sqrt{3}}{\sqrt{2}}x_1^2 \\ \frac{\sqrt{3}}{\sqrt{2}}x_2^2 & 2-x_2 \end{pmatrix}$,

$K_2(x,y) = \exp(\lambda_2 \begin{pmatrix} x_1^2 & y_1^2 \\ x_2^2 & y_2^2 \end{pmatrix})$, $H_2(y,f(y)) = \frac{\theta_2 f^t(y)^2}{1+\|f^t(y)\|^2} \begin{pmatrix} 0 & y_1 \\ y_2 & 0 \end{pmatrix}$,

$h_2(x) = \begin{pmatrix} x_1^2 & x_1 \\ x_1 & x_2^2 \end{pmatrix})$, *and* $G(x,y,z) = \|x\|^\alpha yz$, *for positive* α, *and the norms of vectors and matrices are the euclidean ones. Then a sufficient* *condition for existence and uniqueness of a solution to this integral equation is*

$$\sqrt{2}\left(R_2 e^{2|\lambda_1|}|\theta_1| + R_1 e^{2|\lambda_2|}|\theta_2|\right) \cdot \left(1+\frac{R^2}{1+R^2}\right)\left(\frac{R+(R\wedge(\sqrt{R^2+1}-R))}{1+(R\wedge(\sqrt{R^2+1}-R))^2}\right) < 1$$

where $R_1 = (4+\sqrt{2}|\theta_1|e^{2|\lambda_1|})$, $R_2 = (2+\sqrt{2}|\theta_2|e^{2|\lambda_2|})$ *and* $R = R_1R_2$. *Thus, the set of parameters* $(\lambda_1, \lambda_2, \theta_1, \theta_2)$ *for which the solution is unique contains an unbounded open neigbourhood of the origin.*

Proof : The algebra $M_{2\times 2}(\mathbb{R})$ with usual operations and euclidean norm is complete and satisfies $\|xy\| \leq \|x\|\|y\|$. We have the following inequalities:

(1) $\forall x \in [0,1]^2 \quad \|h_1(x)\| \leq 4, \quad \forall x \in [0,1]^2 \quad \|h_2(x)\| \leq 2,$

(2) $\|K_1(x,y)\| \leq exp(|\lambda_1| \, \| \begin{pmatrix} x_1 & y_1 \\ y_1 & x_2 \end{pmatrix} \|)$, and $\| \int_D \|K_1(x,y)\|dy\|_\infty$
$\leq \sup_{x\in[0,1]^2} \int_{[0,1]^2} e^{|\lambda_1|\sqrt{x_1^2+x_2^2+y_1^2+y_2^2}} dy \leq e^{2|\lambda_1|}$

(3) $\|K_2(x,y)\| \leq \exp(|\lambda_2| \, \| \begin{pmatrix} x_1^2 & y_1^2 \\ y_1^2 & x_2^2 \end{pmatrix} \|)$, and $\| \int_D \|K_2(x,y)\|dy\|_\infty$
$\leq \sup_{x\in[0,1]^2} \int_{[0,1]^2} e^{|\lambda_2|\sqrt{x_1^4+x_2^4+y_1^4+y_2^4}} dy \leq e^{2|\lambda_2|}$

(4) $\|H_1(y,f(y))\| \leq |\theta_1| \frac{\|f(y)^2\|}{1+\|f(y)\|^2}\sqrt{y_1^2+y_2^2}, \quad \|H_2(y,f(y))\|$
$\leq |\theta_2| \frac{\|f^t(y)^2\|}{1+\|f^t(y)\|^2}\sqrt{y_1^2+y_2^2}$ and
$\sup\{\|H_1(y,f(y))\| : y \in D, z \in Im(G)\}$
$\leq \sup\{\|H_1(y,f(y))\| : y \in D, z \in \mathcal{A})\} \leq \sqrt{2}|\theta_1|.$
$\sup\{\|H_2(y,f(y))\| : y \in D, z \in Im(G)\}$
$\leq \sup\{\|H_2(y,f(y))\| : y \in D, z \in \mathcal{A})\} \leq \sqrt{2}|\theta_2|.$

Thus, $\kappa_1 \leq e^{2|\lambda_1|}$, $\kappa_2 \leq e^{2|\lambda_2|}$, $\nu_1 \leq \sqrt{2}|\theta_1|$, $\nu_2 \leq \sqrt{2}|\theta_2|$, and $Im(h_1) \subset B[0,4]$ as well as $Im(h_2) \subset B[0,2]$.

Now, $\|G(x,y_1,z_1) - G(x,y_2,z_2)\| = \|x\|^\alpha\|y_1z_1 - y_2z_2\| = \|x\|^\alpha\|y_1z_1 - y_2z_1 + y_2z_1 - y_2z_2\| \leq \|x\|^\alpha(\|y_1-y_2\|\|z_1\| + \|y_2\|\|z_1-z_2\|)$.

Thus, $\forall x \in D, \forall y_1, y_2 \in Im(h_1) + B[0, \nu_1\kappa_1] \subset B[0, 4 + \sqrt{2}|\theta_1|e^{2|\lambda_1|}]$ $\forall z_1, z_2 \in Im(h_2) + B[0, \nu_2\kappa_2] \subset B[0, 2 + \sqrt{2}|\theta_2|e^{2|\lambda_2|}]$ we have

$$\|G(x, y_1, z_1) - G(x, y_2, z_2)\| \leq (2 + \sqrt{2}|\theta_2|e^{2|\lambda_2|})\|y_2 - y_1\| + (4 + \sqrt{2}|\theta_1|e^{2|\lambda_1|})\|z_2 - z_1\|$$

and $\mu_1 \leq 2 + \sqrt{2}|\theta_2|e^{2|\lambda_2|}$ and $\mu_1 \leq 4 + \sqrt{2}|\theta_1|e^{2|\lambda_1|}$.

$$\begin{aligned} G = \|x\|^\alpha yz \longrightarrow Im(G) \subset B[0, \sup\{\|x\|^\alpha yz : x \in D,\\ y \in B[0, 4 + \sqrt{2}|\theta_1|e^{2|\lambda_1|}], z \in B[0, 2 + \sqrt{2}|\theta_2|e^{2|\lambda_2|}]\}]\\ = B[0, (4 + \sqrt{2}|\theta_1|e^{2|\lambda_1|})(2 + \sqrt{2}|\theta_2|e^{2|\lambda_2|})]. \end{aligned}$$

$$\begin{aligned} &\|H_1(x, y) - H_1(x, z)\| = \|\frac{\theta_1 y^2}{1 + \|y\|^2} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} - \frac{\theta_1 z^2}{1 + \|z\|^2} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\| \\ &\leq \sqrt{2}|\theta_1| \frac{\|y^2(1 + \|z^2\|) - z^2(1 + \|y^2\|)\|}{(1 + \|z\|^2)(1 + \|y\|^2)} \\ &\leq \sqrt{2}|\theta_1|(\|\frac{y^2 - z^2}{1 + \|y\|^2}\| + \|\frac{(\|z^2\| - \|y^2\|)z^2}{(1 + \|z\|^2)(1 + \|y\|^2)}\|) \\ &\leq \sqrt{2}|\theta_1|(\frac{(\|y\| + \|z\|)(\|y - z\|)}{1 + \|y\|^2} + \frac{\|z^2\|(\|y\| + \|z\|)(\|y - z\|)}{(1 + \|z\|^2)(1 + \|y\|^2)} \\ &= \sqrt{2}|\theta_1|(1 + \frac{\|z^2\|}{1 + \|z\|^2})(\frac{\|y\| + \|z\|}{1 + \|y\|^2})\|y - z\| \end{aligned}$$

Now, the maximization of $g(u, v) = \left(1 + \frac{u^2}{1+u^2}\right)\left(\frac{u+v}{1+v^2}\right)$ subjected to the constraint $(u, v) \in [0, R]^2$, for arbitrary R, furnishes $u = R$ and $v = R \wedge (\sqrt{R^2 + 1} - R)$ so that, letting $R = (4 + \sqrt{2}|\theta_1|e^{2|\lambda_1|})(2 + \sqrt{2}|\theta_2|e^{2|\lambda_2|})$ we have

$$\iota_1 \leq \sqrt{2}|\theta_1| \left(1 + \frac{R^2}{1 + R^2}\right) \left(\frac{R + (R \wedge (\sqrt{R^2 + 1} - R))}{1 + (R \wedge (\sqrt{R^2 + 1} - R))^2}\right)$$

Analogously, since $z \in B[0, r] \longleftrightarrow z^t \in B[0, r]$, we have

$$\|H_2(x, y) - H_2(x, z)\| \leq \sqrt{2}|\theta_2| \left(1 + \frac{\|(z^t)^2\|}{1 + \|z^t\|^2}\right) \left(\frac{\|y^t\| + \|z^t\|}{1 + \|y^t\|^2}\right) \|y^t - z^t\| \leq$$

$$\sqrt{2}|\theta_2|\left(1+\frac{R^2}{1+R^2}\right)\left(\frac{R+(R\wedge(\sqrt{R^2+1}-R))}{1+(R\wedge(\sqrt{R^2+1}-R))^2}\right)\|y-z\|$$

and

$$\iota_2 \leq \sqrt{2}|\theta_2|\left(1+\frac{R^2}{1+R^2}\right)\left(\frac{R+(R\wedge(\sqrt{R^2+1}-R))}{1+(R\wedge(\sqrt{R^2+1}-R))^2}\right).$$

In this way, by Theorem 2.1, existence and uniqueness of solution of the integral equation is implied by $\mu_1\kappa_1\iota_1+\mu_2\kappa_2\iota_2<1$ and, consequently, by :

$$\begin{aligned} s(\theta_1,\theta_2,\lambda_1,\lambda_2) := {} & \sqrt{2}((2+\sqrt{2}|\theta_2|e^{2|\lambda_2|}])e^{2|\lambda_1|}|\theta_1| \\ & + (4+\sqrt{2}|\theta_1|e^{2|\lambda_1|})e^{2|\lambda_2|}|\theta_2|)\cdot(1+\frac{R^2}{1+R^2}) \\ & \cdot(\frac{R+(R\wedge(\sqrt{R^2+1}-R))}{1+(R\wedge(\sqrt{R^2+1}-R))^2}) < 1 \end{aligned}$$

Now, observe that s is a continuous function and $s^{-1}([0,1)) \supset \{(0,0)\}\times\mathbb{R}^2$. Finally, we remark that one can consider the situation where either K_i or H_i takes values in the field instead of in the algebra and obtain variants of the theorems presented thus far.

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