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Necessary and Sufficient conditions for Existence of Solutions of a Divergence-type Variational Problem

José Matias

Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1, 1049-001 Lisboa, Portugal

E-mail address: `jmatias@math.ist.utl.pt`

Abstract. We look for necessary and sufficient conditions for the existence of solutions of the minimization problem

$$(P) \qquad \inf\left\{\int_\Omega f(PDu(x))\,dx : u \in u_{\zeta_0} + W_0^{m,\infty}(\Omega;\mathbb{R}^N)\right\}$$

where PD is a particular type of differential operator of order m, (which we identify as of divergence type) and the boundary data u_{ζ_0} satisfes $PDu_{\zeta_0}(x) = \zeta_0$ for $\zeta_0 \in \mathbb{R}$ given.

1. **Introduction**

The search for minimizers of

$$\inf\left\{\int_\Omega f(\nabla u(x))\,dx : u \in u_0 + W_0^{1,\infty}(\Omega)\right\}$$

when the integrand function f is non convex, has been undertaken extensively (see, for example, [3], [4], [8], [9] and the references therein). Dacorogna and Marcellini ([8]) showed that a necessary condition for existence of solutions to this problem is that the convex envelope of f, f^{**}, is globally affine.

This work follows closely [1] where the problem is treated in the general setting of differential forms and [2] where the problem was treated in the case of the curl operator.

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However, the (simpler) case of divergence was not explicitely treated. In this work, we deal with the problem for a particular type of differential operators, that namely comprise the divergence operator.

In particular, combining this work with the results of [1], [7] and [2], for du a differential form of order k in $\mathbb{R}^3$, $(0 \leq k \leq 2)$, the problem of finding necessary and sufficient conditions for existence of solutions of

$$(P) \qquad \inf\left\{\int_\Omega f(du(x))\,dx : u \in u_{\zeta_0} + W_0^{1,\infty}(\Omega;\mathbb{R}^{\binom{n}{k}})\right\}$$

where the boundary data u_{ζ_0} satisfies $du_{\zeta_0} = \zeta_0$, for ζ_0 a given vector in $\mathbb{R}^{\binom{n}{k}}$, becomes completely solved.

2. Preliminaries

We start with some notations which are used throughout this paper. Although these notations are somewhat standard we mention them here for the sake of completeness.

- $\mathbb{R}_0^+$ denotes the set of all non-negative real numbers.
- For $E \subseteq \mathbb{R}^N, E \neq \emptyset$, we write span$E$ to denote the subspace spanned by E.
- Let W be a subspace of $\mathbb{R}^N$. We write dim$W$ to denote the dimension of W.
- H^k denotes the k-dimensional Hausdorff measure.
- $B(\Omega)$ denotes the Borel σ-algebra of subsets of Ω.
- co U denotes the convex hull of $U \subseteq \mathbb{R}^N$ and $\overline{co}U$ its closure.
- For a function $f : \mathbb{R}^N \to \mathbb{R}$, f^{**} denotes the convex envelope of f, that is,
$$f^{**} = \inf\{g : g \text{ convex},\ g \leq f\}.$$
- Ω denotes an open bounded subset of $\mathbb{R}^n$, and we denote its Lebesgue measure by meas(Ω).
- We denote by $B_n(x,\epsilon)$ the open ball in $\mathbb{R}^n$ centered at x with radius ϵ.
- the letter C will be used throughout this work to indicate a constant whose value might change fro line to line.
- We use the standard multi-index notation: for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{N},\ j = 1, \ldots, n, |\alpha| = \alpha_1 + \ldots + \alpha_n$, and for $u : \mathbb{R}^n \to \mathbb{R}^N,\ \partial^\alpha u_i$

denotes the partial derivative

$$\partial^\alpha u_i = \frac{\partial^{|\alpha|} u_i}{dx_1^{\alpha_1} dx_2^{\alpha_2} \dots dx_n^{\alpha_n}}.$$

3. Statement of the problem

For PD as above and for $f : \mathbb{R} \to \mathbb{R}$ a continuous function, we look for necessary and sufficient conditions for existence of solutions of the following minimization problem:

$$(P) \qquad \inf \left\{ \int_\Omega f(PDu(x))\, dx : u \in u_{\zeta_0} + W_0^{m,\infty}(\Omega; \mathbb{R}^N) \right\}$$

where:

a) for $i = 1, \dots, N$, $A_i = \{\alpha,\ 0 < |\alpha| \leq m_i\}$ is a set of multi-indices, where $m_i \in \mathbb{N}_1$ and for $u : \mathbb{R}^n \to \mathbb{R}^N$, PD is the linear partial differential operator given by

$$PDu = \sum_{i=1}^{N} \sum_{\alpha \in A_i} c_{\alpha,i} \partial^\alpha u_i,$$

under one of the following hypothesis:

$$(H_1) \quad \exists i \in \{1, \dots, N\} : |\alpha| = m_i,\ \forall \alpha \in A_i.$$

or

$$(H_2) \quad \exists i \in \{1, \dots, N\}, \exists j \in \{1 \dots, n\} : \alpha_j = \text{constant},\ \forall \alpha \in A_i.$$

b) the boundary data u_{ζ_0} satisfies $PDu_{\zeta_0}(x) = \zeta_0$, for $\zeta_0 \in \mathbb{R}$ given, and $m = \max\{m_i,\ i = 1, \dots, N\}$,

It will be helpfull to consider the auxiliary problem

$$(P^{**}) \qquad \inf \left\{ \int_\Omega f^{**}(PDu(x))\, dx : u \in u_{\zeta_0} + W_0^{m,\infty}(\Omega; \mathbb{R}^N) \right\}$$

where f^{**} is the convex envelpe of f. By convexity of f^{**}, using Jensen's inequality and the density of C_c^∞ in $W_0^{m,\infty}$, it is easy to see that

$$\inf(P^{**}) = f^{**}(\zeta_0)\text{meas}(\Omega).$$

4. The Approximation Lemma

In this section we prove an approximation lemma which will be used to establish suufficient conditions for the existence of solutions of problem (P). This result also allows us to prove a relaxation result which states that $\inf(P) = \inf(P^{**})$.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ open, bounded and let $t \in [0,1]$ and $\zeta, \eta \in \mathbb{R}$. Let $\phi \in W^{m,\infty}(\Omega;\mathbb{R}^N)$ be such that $PD\phi = t\zeta + (1-t)\eta$. Then, for every $\epsilon > 0$, there exist $u \in \phi + W_0^{m,\infty}(\Omega;\mathbb{R}^N)$ and $\Omega_\zeta, \Omega_\eta$, disjoint open subsets of Ω such that*

$$\begin{cases} |\mathrm{meas}(\Omega_\zeta) - t\,\mathrm{meas}(\Omega)| \leq \epsilon \\ |\mathrm{meas}(\Omega_\eta) - (1-t)\mathrm{meas}(\Omega)| \leq \epsilon \\ u = \phi \text{ in a neighbourhood of } \partial\Omega \\ ||u-\phi||_{L^\infty} \leq C\epsilon \\ PDu(x) = \begin{cases} \zeta \text{ in } \Omega_\zeta \\ \eta \text{ in } \Omega_\eta \end{cases} \\ \mathrm{dist}(PDu(x), [\zeta,\eta]) \leq C\epsilon, \text{ a.e. in } \Omega, \end{cases} \tag{4.1}$$

where $[\zeta,\eta]$ denotes the closed interval with endpoints ζ and η.

Proof. We are assuming that $\zeta \neq \eta$, otherwise the result is trivial since it suffices to take $u = \phi$. We follow here the ideas presented in [2].

Without loss of generality we may assume that Ω is the unit cube centered at the origin with its faces parallel to the coordinate axes. Indeed, if this is not the case, we can express Ω as the disjoint union of cubes whose faces are parallel to the coordinate axes plus a set of small measure; in this case a solution u for (4.1) with respect to Ω can be constructed from a solution of (4.1) when Ω is the unit cube by setting $u = \phi$ on the set of small measure and by using homothetics and translations in each of the small subcubes.

Let $\epsilon > 0$, let $\Omega_\epsilon \subset\subset \Omega$ and let $h \in C_0^\infty(\Omega)$ and $L = L(\Omega)$ be such that

$$\begin{cases} \text{meas }(\Omega\backslash\Omega_\epsilon) \le \frac{\epsilon}{2} \\ 0 \le h(x) \le 1, \ \forall x \in \Omega \\ h(x) = 1, \ \forall x \in \Omega_\epsilon \\ |D^k h(x)| \le L\epsilon^{-k}, \ \forall x \in \Omega\backslash\Omega_\epsilon, \ \forall 0 < k \le m \end{cases}$$

Let $\delta > 0$. Suppose first that PD satisfies (H_1) and that $N = 1$. Let A be the set of all multi-indices interveening in PD, and let $\bar{\alpha_1} = \max\{\alpha_1, \alpha \in A\}$. Select one $\bar{\alpha} \in A$ such that its first component equals $\bar{\alpha_1}$ (there could be more than one). Notice that w.l.o.g. we are assuming that $\bar{\alpha_1} > 0$ (otherwise, take other variable involved). By a standard procedure we may construct a C^∞ function $g : [0,1] \to \mathbb{R}$ and sets I_ζ, I_η which are unions of dispoint open subintervals of $[0,1]$, so that

$$\begin{cases} g^{\bar{\alpha_1}}(x_1) = \begin{cases} 1-t & \text{if } x_1 \in I_\zeta \\ -t & \text{if } x_1 \in I_\eta \end{cases} \\ g^{\bar{\alpha_1}}(x_1) \in [-t, 1-t], \ \forall x_1 \in [0,1], \\ |\text{meas}(I_\zeta) - t| \le \frac{\epsilon}{2}, \\ |\text{meas}(I_\eta) - (1-t)| \le \frac{\epsilon}{2}, \\ |g(x_1)| \le \delta, |g'(x_1)) \le \delta, \ldots, |g^{\bar{\alpha_1}-1}(x_1)| \le \delta \ \forall x_1 \in [0,1] \end{cases}$$

Let

$$\Omega_\zeta := \{x \in \Omega_\epsilon : x_1 \in I_\zeta\},$$

and

$$\Omega_\eta := \{x \in \Omega_\epsilon : x_1 \in I_\eta\},.$$

We now define the function $w \in C^\infty(\bar{\Omega})$ by

$$w(x) = \frac{1}{C_{\bar{\alpha}}(\alpha_2! \ldots \alpha_n!)} g(x_1) x_2^{\bar{\alpha_2}} \ldots x_n^{\bar{\alpha_n}} (\zeta - \eta).$$

If $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N, \ N > 1$, the same process works with the necessary adaptations. In fact, we just need to construct w_1 as before (i.e. like w in the case $N = 1$, regarding the component u_1 of u) and then set;

$$w = (w_1, 0, \ldots, 0).$$

(where, once again, we assumed w.l.o.g. that the component u_1 is involved by the operator PD.)

We claim that the function $u : \Omega \to \mathbb{R}$ given by

$$u = \phi + hw$$

satisfies the properties listed in (4.1). Indeed, since $h \equiv 1$ in Ω_ϵ, we have

$$\begin{aligned} PD(u) &= PD\phi + PD(hw) \\ &= t\zeta + (1-t)\eta + g^{\bar{\alpha_1}}(x_1)(\zeta - \eta) \end{aligned}$$

so that

$$PDu(x) = \begin{cases} \zeta & \text{in } \Omega_\zeta \\ \eta & \text{in } \Omega_\eta \end{cases}$$

Using the fact that $h \leq 1$ and $g(x_1)| \leq \delta$, it is possible to choose δ sufficiently small with respect to ϵ so that $||u - \phi||_\infty = ||hw||_\infty < \epsilon$.

On he other hand,

$$\begin{aligned} PDu &= PD\phi + PD(hw) \\ &= t\zeta + (1-t)\eta + hg^{\bar{\alpha_1}}(x_1)(\zeta - \eta) + R(h,\alpha) \qquad (4.2) \\ &= (t + hg^{\bar{\alpha_1}}(x_1))\zeta + (1 - (t + hg^{\bar{\alpha_1}}(x_1)))\eta + R(h,\alpha), \end{aligned}$$

where, using the estimates on the derivatives of h and g, we can once again choose δ sufficiently small with respect to ϵ, in order to obtain

$$|R(h,\alpha)| \leq \epsilon. \qquad (4.3)$$

Since $0 \leq t + hg^{\bar{\alpha_1}}(x_1) \leq 1$, by (4.2) and (4.3) we conclude that

$$\text{dist } (PDu(x), [\zeta, \eta]) \leq |R(h,\alpha)| \leq \epsilon, \text{ a.e. in } \Omega.$$

The proof of the remaining statements f (4.1) is straightforward.

Suppose now that PD satisifes (H_2). The process is similar, just constructing w using one of the lowest order derivatives. □

Theorem 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ continuous. Then*

$$\inf(P) = \inf(P^{**}).$$

Proof. The inequality $\inf(P) \geq \inf(P^{**})$ is trivial. We will use the fact that $\inf(P^{**}) = f^{**}(\zeta_0)\text{meas}(\Omega)$ to prove the reverse inequality.

Given $\zeta_0 \in \mathbb{R}$ there exist $t_i > 0$ and $\zeta_i \in \mathbb{R}$ such that $\sum_{i=1}^{2} t_i = 1,\ \zeta_0 = \sum_{i=1}^{2} t_i\zeta_i$ and

$$f^{**}(\zeta_0) = \sum_{i=1}^{2} t_i f(\zeta_i).$$

Applying the previous lemma to ζ_0 and u_{ζ_0} we obtain a sequence $u_n \in u_{\zeta_0} + W_0^{m,\infty}(\Omega)$, and $\Omega_i, i = 1, 2$, disjoint open subsets of Ω such that

$$\begin{cases} |\text{meas}(\Omega_i) - t_i\text{meas}(\Omega)| \leq \frac{1}{n},\ i = 1, 2, \\ \\ ||u_n - u_{\zeta_0}||_\infty \leq \frac{C}{n} \\ \\ PDu_n(x) = \zeta_i \text{ in } \Omega_i, i = 1, 2 \\ \\ \text{dist}(PDu_n(x),\ \text{co}\{\zeta_1, \zeta_2\}) \leq \frac{C}{n},\ \text{a.e. in } \Omega. \end{cases}$$

The sequence u_n is admissible for problem (P), so we have

$$\begin{aligned} &\inf(P) \leq \lim_{n\to\infty} \textstyle\int_\Omega f(PDu_n(x))\,dx \\ &= \lim_{n\to\infty} \left(\textstyle\int_{\Omega\setminus\cup_{i=1}^2\Omega_i} f(PDu_n(x))\,dx + \sum_{i=1}^2 \int_{\Omega_i} f(\zeta_i)\,dx \right) \\ &\leq \lim_{n\to\infty} \left(\textstyle\int_{\Omega\setminus\cup_{i=1}^2\Omega_i} f(PDu_n(x))\,dx + \sum_{i=1}^2 f(\zeta_i)t_i\text{meas}(\Omega) + \frac{C}{n} \right) \\ &= f^{**}(\zeta_0)\text{meas}(\Omega) = \inf(P^{**}), \end{aligned}$$

where

$$\lim_{n\to\infty} \int_{\Omega\setminus\cup_{i=1}^2\Omega_i} f(PDu_n(x))\,dx = 0$$

since $\text{meas}(\Omega\setminus \cup_{i=1}^2 \Omega_i) \leq \frac{2}{n}$ and by the continuity of f, the sequence $(f(PDu_n(.)))$ is uniformly bounded. □

5. Necessary Conditions

We look for necessary conditions for the existence of solutions of the problem

$$(P) \qquad \inf\left\{ \int_\Omega f(PDu(x))\,dx : u \in u_{\zeta_0} + W^{m,\infty}(\Omega; \mathbb{R}^N) \right\}$$

where the boundary data u_{ζ_0} satisfies $PDU_{\zeta_0}(x) = \zeta_0$, for $\zeta_0 \in \mathbb{R}$ given and $f : \mathbb{R} \to \mathbb{R}$ is continuous.

The set

$$K := \{\zeta \in \mathbb{R} : f^{**}(\zeta) < f(\zeta)\}$$

will play an important role in this analysis.

Proposition 5.1. *Suppose u solves (P). Then u is a solution of*

$$(I) \qquad \begin{cases} f(PDu) = f^{**}(PDu) & \text{a.e. in } \Omega \\ \\ u = u_{\zeta_0} & \text{on } \partial\Omega \end{cases}$$

Proof. If u is a solution of (P) then clearly $u = u_{\zeta_0}$ on $\partial\Omega$. On the other hand, by Theorem (4.1), we know that

$$\inf(P) = \inf(P^{**}) = f^{**}(\zeta_0)\text{meas}(\Omega).$$

Hence, by convexity of f^{**}, Jensen's inequality and by density, we have that

$$f^{**}(\zeta_0)\text{meas}(\Omega) = \inf(P) = \int_\Omega f(PDu(x))\,dx$$

$$\geq \int_\Omega f^{**}(PDu_n(x))\,dx \geq f^{**}\left(\frac{1}{\text{meas}(\Omega)}\int_\Omega PDu(x)\,dx\right)\text{meas}(\Omega)$$

$$= f^{**}\left(\frac{1}{\text{meas}(\Omega)}\int_\Omega \zeta_0 + PD(u(x) - u_{\zeta_0}(x))\,dx\right)\text{meas}(\Omega)$$

$$= f^{**}(\zeta_0)\text{meas}(\Omega).$$

It follows that $f(PDU(x)) = f^{**}(PDu(x))$ for a.e. $x \in \Omega$. $\square$

Remark 5.1. *When ζ_0 is a vector (for instance in the case of the gradient or the curl) there is another necessary condition related to the existence of directions of strict convexity for f^{**} at ζ_0. However, in this simpler case, f^{**} is affine in each connected component of K.*

6. **Sufficient Conditions**

Recall that

$$K = \{\zeta \in \mathbb{R} : f^{**}(\zeta) < f(\zeta)\},$$

and that the boundary data u_{ζ_0} satisfies $PDu_{\zeta_0} = \zeta_0$. Our main result in this section is the following

Theorem 6.1. *Let $K \subset \mathbb{R}$ be bounded and connected and let $\zeta_0 \in K$. Then, if $u \in W^{m,\infty}(\Omega;\mathbb{R}^N)$ is a solution of*

$$(I_1) \qquad \begin{cases} PDu \in \partial K \\ \\ u = u_{\zeta_0} \text{ on } \partial\Omega \end{cases}$$

then u is also a solution of (P).

Proof. Since f^{**} is affine in $\bar{K}$ at ζ_0, if u is a solution of (I_1), then there exist $a, b \in \mathbb{R}$, such that $f^{**}(PDu) = aPDu + b$. As $PDu \in \partial K$, we have that

$$\begin{aligned} \textstyle\int_\Omega f(PDu(x))\,dx &= \textstyle\int_\Omega f^{**}(PDu(x))\,dx \\ &= \textstyle\int_\Omega aPDu(x) + b\,dx \end{aligned} \tag{6.4}$$

Since

$$\int_\Omega a(u(x) - u_{\zeta_0}(x))\,dx = 0,$$

and hence

$$\int_\Omega aPDu(x)\,dx = \int_\Omega a\zeta_0\,dx,$$

we conclude that

$$\begin{aligned} \textstyle\int_\Omega f(PDu(x))\,dx &= \textstyle\int_\Omega a\zeta_0 + b\,dx \\ &= f^{**}(\zeta_0)\mathrm{meas}(\Omega) \\ &= \inf(P^{**}) = \inf(P) \end{aligned}$$

that is, u is a non-trivial solution of (P). Notice that, since $\zeta_0 \in \partial K, u_{\zeta_0}$ is not a solution of (I_1), and so, by Proposition (4.1), it is not a solution of (P) either. If f^{**} is globally affine the equalities (6.4) hold trivially and so the result follows as in the first case.

$\square$

Before proving existence of solutions to problem (I_1), we need the following definitions:

Definition 6.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. For $\theta > 0$, let W_θ be the set of all functions $u \in C^m_{\mathrm{piec}}(\bar{\Omega}; \mathbb{R}^N)$ for which there exists $\Omega_\theta \subset \Omega$ such that $\mathrm{meas}(\Omega \backslash \Omega_\theta) \leq \theta$, and PDu is piecewise constant in Ω_θ.*

Definition 6.2. *Let $E; S \subseteq \mathbb{R}$. We say that S has the relaxation property with respect to E, if, for every bounded open set $\Omega \subseteq \mathbb{R}^n$ and for every map u_ζ satisfying $PDu_\zeta = \zeta \in \mathrm{int}S$, there exists a sequence $u : n \in W_{\frac{1}{n}}$ such* *that*

$$\begin{cases} u_n \in u_\zeta + W_0^{m,\infty}(\Omega;\mathbb{R}^N) \\ u_n \to u_\zeta \text{ in } L^\infty(\Omega;\mathbb{R}^N),\ PDu_n \overset{*}{\rightharpoonup} \zeta \text{ in } L^\infty(\Omega) \\ PDu_n(x) \in E \cup \mathrm{int}S \text{ a.e. in } \Omega \\ \lim_{n\to+\infty} \int_\Omega \mathrm{dist}(PDu_n(x), E)\, dx = 0 \end{cases}$$

For the proof of existence of solutions to problem (I_1), we will need to show that $\mathrm{co}(\partial K)$ has the relaxation property with respect to ∂K. The following lemma provides a way of doing this.

Lemma 6.1. *Let $E, E_\delta \subseteq \mathbb{R}$, for $\delta \in]0,\delta_0[$, be compact sets such that*

i) $\mathrm{co}E_\delta \subset \mathrm{intco}E$ *for every* $\delta \in]0,\delta_0[$;
ii) $\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0$, *such that,* $\forall \delta \in]0,\delta(\epsilon)[, \rho \in E_\delta \Rightarrow \mathrm{dist}(\rho, E) \le \epsilon$;
iii) *if* $\rho \in \mathrm{intco}E$, *then* $\rho \in \mathrm{co}E_\delta$, *for every* $\delta > 0$ *sufficiently small.*

Then $\mathrm{co}E$ *has the relaxation property with respect to* E.

Proof. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $u : \Omega \to \mathbb{R}^N$ be a function satisfying

$$PDu = \zeta \in \mathrm{intco}\, E.$$

We claim that there exists a sequence $u_n \in W_{\frac{1}{n}}$ such that

$$\begin{cases} u_n \in u + W_0^{m,\infty}(\Omega;\mathbb{R}^N) \\ u_n \to u \text{ in } L^\infty(\Omega;\mathbb{R}^N),\ PDu_n \overset{*}{\rightharpoonup} \zeta \text{ in } L^\infty(\Omega) \\ PDu_n(x) \in E \cup \mathrm{intco}E \text{ a.e. in } \Omega \\ \lim_{n\to+\infty} \int_\Omega \mathrm{dist}(PDu_n(x), E)\, dx = 0 \end{cases}$$

Fix $\epsilon > 0$ and let $\delta = \delta(\epsilon)$ be determined according to *ii*). By *iii*), we may find $\delta_1 < \delta$ such that

$$\zeta \in \mathrm{intco}E \Rightarrow \zeta \in \mathrm{co}E_{\delta_1}.$$

Therefore we may write $\zeta = t\zeta_1 + (1-t)\zeta_2$ with $\zeta_1, \zeta_2 \in E_{\delta_1}$. By property *i*), we can choose $\epsilon' \in]0,\epsilon[$ such that the ϵ'-neighbourhood of $\mathrm{co}E_{\delta_1}$ is contained in $\mathrm{intco}E$. Using the Approximation Lemma (4.1) we may find

$u_\epsilon \in C^m_{\text{piec}}(\bar{\Omega};\mathbb{R}^N)$ and Ω_1, Ω_2 disjoint open subsets of Ω, such that

$$\begin{cases} \text{meas}(\Omega\backslash(\Omega_1\cup\Omega_2)) = O(\epsilon') \\ u_\epsilon = u \text{ near } \partial\Omega \\ ||u_\epsilon - u||_\infty \leq C\epsilon' \\ PDu_\epsilon(x) = \begin{cases} \zeta_1 \text{ in } \Omega_1 \\ \zeta_2 \text{ in } \Omega_2 \end{cases} \\ \text{dist}(PDu_\epsilon(x),\ \text{co}E_{\delta_1}) \leq C\epsilon',\ \text{a.e in } \Omega \end{cases}$$

Now, the choice of ϵ' ensures that

$$PDu_\epsilon \in \ \text{co}E,$$

and in view of ii), taking into account that $\text{dist}(PDu_\epsilon, E)$ is a bounded function, we conclude that

$$\begin{aligned} \int_\Omega \text{dist}(PDu_\epsilon(x), E)\ dx \ &= \int_{\Omega_1} \text{dist}(\zeta_1, E)\ dx + \int_{\Omega_2} \text{dist}(\zeta_2, E)\ dx \\ &+ \int_{\Omega\backslash(\Omega_1\cup\Omega_2)} \text{dist}(PDu_\epsilon(x)\ dx \\ &\leq \epsilon\text{meas}(\Omega_1) + \epsilon\text{meas}(\Omega_2) + O(\epsilon') = O(\epsilon), \end{aligned}$$

so that the claim is obtained by letting $\epsilon \to 0^+$. Finally, we need to check if $u_n \in W_{\frac{1}{n}}$. Letting $2\epsilon' = \frac{1}{n}$, we have that $(u_n) \subset C^m_{\text{piec}}(\bar{\Omega};\mathbb{R}^N)$ and

$$PDu_n(x) = \begin{cases} \zeta_1 \text{ in } \Omega_1 \\ \zeta_2 \text{ in } \Omega_2 \end{cases}$$

where $\text{meas}(\Omega\backslash(\Omega_1\cup\Omega_2)) \leq \frac{1}{n}$. Thus, $u_n \in W_{\frac{1}{n}}$. □

In what follows, we will need the following result of convex analysis, which can be found in [11].

Proposition 6.1. *Let $E \subset \mathbb{R}$ be compact and such that* $\text{intco}E \neq \emptyset$. *Let E_{ext} denote the set of extreme points of* co$E$. *Then, there exists a convex a lower semicontinuous function $\psi : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ such that*

$$E_{\text{ext}} = \{\zeta \in \mathbb{R} : \psi(\zeta) = 0\}$$

and

$$\text{co}E = \ \text{co}E_{\text{ext}} = \{\zeta \in \mathbb{R} : \psi(\zeta) \leq 0\}.$$

Finally we are in position to state the following

Theorem 6.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $E \subset \mathbb{R}$ be a compact set. Let $\phi \in C^m_{\text{piec}}(\bar{\Omega}; \mathbb{R}^N)$ be such that $PD\phi$ is piecewise constant in Ω an $PD\phi \in E \cup \text{intco}E$. Then there exists $u \in C(\bar{\Omega}; \mathbb{R}^N)$ with $PDu \in L^\infty(\Omega)$ and satisfying*

$$\begin{cases} PDu(x) \in E \text{ a.e. } x \in \Omega \\ \\ u(x) = \phi(x),\ \forall x \in \partial\Omega \end{cases}$$

Proof. Assume w.l.o.g. that Ω is bounded and that $\phi \in C^m(\bar{\Omega}); \mathbb{R}^N)$. Assume also that $\text{intco}E \neq \emptyset$ since otherwise the result is trivial (it suffices to take $u \equiv \phi$).

Step 1

Under the previous assumptions we show that co$E$ has the relaxation property with respect to E. For this purpose we use Lemma (6.1). Choose $\alpha_0 \in$ intco$E$ and, for $\delta \in]0,1[$, define the sets

$$E_\delta := \delta\alpha_0 + (1-\delta)E.$$

Notice that these sets are compact since E is compact. If $\rho \in E_\delta$ then $\rho = \delta\alpha_0 + (1-\delta)\bar{\rho}$, with $\bar{\rho} \in E$, and hence

$$\text{dist}(\rho, E) \leq |\rho - \bar{\rho}| < \epsilon,$$

provided we take $\delta(\epsilon) = \frac{\epsilon}{\text{diamco}E}$. This proves property *ii*). As for property *iii*), if $\rho \in$ intco$E$, since

$$\lim_{\delta \to 0} \left| \left(\frac{\rho}{1-\delta} - \frac{\delta}{1-\delta}\alpha_0 \right) - \rho \right| = 0,$$

it follows that

$$\frac{\rho}{1-\delta} - \frac{\delta}{1-\delta}\alpha_0 \in \text{ co}E$$

for $\delta > 0$ sufficiently small, and so $\rho \in \delta\alpha_0 + (1-\delta)\text{co}E = \text{ co}E_\delta$.

It remains to show property *i*). Since $\alpha_0 \in$ intco$E$ there exists $r > 0$ such that

$$B_1(\alpha_0, r) \subset \text{ co}E. \tag{6.5}$$

Therefore, it suffices to show that

$$B_1(\delta\alpha_0 + (1-\delta)\rho, \delta r) \subset \text{ co}E,\ \forall \rho \in \text{ co}E.$$

Let $x \in B_1(\delta\alpha_0 + (1-\delta)\rho, \delta r)$ and let

$$\alpha := \frac{x - (1-\delta)\rho}{\delta}.$$

Notice that $|\delta\alpha - \delta\alpha_0| < \delta r$ and so, from (6.5), we have that $\alpha \in \text{co}E$, and we conclude that
$$x = \delta\alpha + (1-\delta)\rho \in \text{co}E.$$

Step 2

Assume first that $PD\phi(x) \in \text{intco}E$ for a.e. $x \in \Omega$. Since E is compact and $\text{intco}E \neq \emptyset$, by Proposition (6.1) applied to E_{ext}, we conclude that there exists a convex and lower semicontinuous function
$$\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\},$$
such that
$$E = E_{\text{ext}} = \{\zeta \in \mathbb{R} : \psi(\zeta) = 0\}, \tag{6.6}$$
and
$$\text{co}E = \text{co}E_{\text{ext}} = \{\zeta \in \mathbb{R} : \psi(\zeta) \leq 0\}. \tag{6.7}$$
Let V be the set of fuctions $u : \Omega \to \mathbb{R}^N$ so that there exists $u_n \in W_{\frac{1}{n}}$ satisfying $u_n = \phi$ on $\partial\Omega, PDu_n \in E \cup \text{intco}E$ a.e. in Ω, and $u_n \to u$ in $L^\infty(\Omega;\mathbb{R}^N)$. Notice that $\phi \in V$ and V is a complete metric space when endowed with the C^0 norm.

The compactness of E and $\text{co}E$, the convexity and lower semicontinuity of ψ and (6.6), (6.7) yield
$$\begin{aligned} &V \subset \{u \in C(\bar{\Omega};\mathbb{R}^N) : PDu \in L^\infty(\Omega),\\ &u = \phi \text{ on } \partial\Omega, \psi(PDu(x)) \leq 0 \text{ a.e. } x \in \Omega\}. \end{aligned} \tag{6.8}$$
For $u \in V$, set
$$L(u) := \int_\Omega \psi(PDu(x))\, dx.$$
From the lower semicontinuity and convexity of ψ we have that, for every $u \in V$,
$$L(u) \leq \liminf_{u_n \to, u_n \in V} L(u_n). \tag{6.9}$$
Also, by (6.8), $L(u) \leq 0$ and
$$L(u) = 0 \Leftrightarrow PDu(x) \in E \; for \text{ a.e. } x \in \Omega. \tag{6.10}$$
Define
$$V^k := \left\{ u \in V : L(u) > -\frac{1}{k} \right\}.$$
By (6.9), V^k is open in V. We will now prove that V^k is dense in V, in which case it will follow from Baire's Category Theorem that $\bigcap_k V^k$ is dense in v. In particular,
$$\bigcap_k V^k = \{u \in V : L(u) = 0\} \neq \emptyset.$$

Thus, there exists $u \in V$ such that $L(u) = 0$, that is, by (6.10), such that $PDu(x) \in E = E_{\text{ext}}$ for a.e. $x \in \Omega$, and since $u \in V$ we are done.

Therefore, it remains to prove the density result, i.e., that for fixed $k \in \mathbb{N}, u \in V$, and $\epsilon \in]0, \frac{1}{k}[$ sufficiently small, we can find $u_\epsilon \in V^k$ such that $||u_\epsilon - u||_\infty \leq \epsilon$. We will prove this property under the further assumption that, for some $\theta > 0$, small, $u \in W_\theta$ and $PDu(x) \in E \cap \text{intco}E$ for a.e. $x \in \Omega$. The general case will follow from the definition of V. Also, by working on each subset of Ω where u is of class C^m and PDu is constant, and by setting $u_\epsilon = u$ on $\Omega \backslash \Omega_\theta$ we can assume, w.l.o.g., that $u \in C^m(\bar{\Omega}; \mathbb{R}^N), PDu$ is constant in Ω and $PDu(x) \in$ intco$E$ (otherwise the result is trivial).

By compactness of E and co$E$ we have that

$$\zeta \in E \cup \text{ co}E \;\Rightarrow\; \text{dist}(\zeta, E) \leq \beta \tag{6.11}$$

for some $\beta > 0$. By the convexity and lower semicontinuity of ψ and (6.6), we can fix $\delta = \delta(\epsilon) > 0$, such that, for any measurable function $\mu : \mathbb{R}^n \to E \cap \text{ co}E$, the following holds

$$\int_\Omega \text{dist}(\mu(x), E)\, dx \leq \delta \Rightarrow \int_\Omega \psi(\mu(x))\, dx \geq -\epsilon. \tag{6.12}$$

The result now follows immediately from the relaxation property. Indeed, since co$E$ has the relaxation property with respect to E and since $PDu(x)$ is a constant belonging to intco$E$, there exists a sequence $u_\epsilon \in W_\epsilon$ such that

$$\begin{cases} u_\epsilon \in u + W_0^{m,\infty}(\Omega; \mathbb{R}^N) \\ ||u_\epsilon - u||_\infty \leq \epsilon \\ \text{curl } u_\epsilon(x) \in E \cup \text{ intco}E \text{ a.e. } x \in \Omega \\ \int_\Omega \text{dist}(PDu_\epsilon(x), E)\, dx \leq \delta \end{cases}$$

From (6.12) we conclude that

$$\int_\Omega \psi(PDu_\epsilon(x))\, dx \geq -\epsilon \Rightarrow L(u_\epsilon \geq -\epsilon > -\frac{1}{k}$$

for $\epsilon < \frac{1}{k}$. Thus $u_\epsilon \in V^k$ and $||u_\epsilon - u|\_\infty \leq \epsilon$, so the proof of the density result is complete.

Step 3

We now turn to the general case, $PD\phi(x) \in E \cup$ intco$E$ for a.e. $x \in \Omega$, and we let

$$\Omega_0 := \{x \in \Omega : PD\phi(x) \in E\}, \;\; \Omega_1 := \Omega \backslash \Omega_0.$$

Then Ω_1 is open by continuity, and $PD\phi(x) \in$ intco$E$ for a.e. $x \in \Omega_1$. We apply Step 2 to the set Ω_1 to obtain a function $u_1 \in C(\bar{\Omega}_1; \mathbb{R}^N)$ with $PDu_1 \in L^\infty(\Omega_1)$ and such that

$$\begin{cases} PDu_1(x) \in E_{\text{ext}} \text{ for a.e. } x \in \Omega_1 \\ \\ u_1(x) = \phi(x) \text{ on } \partial\Omega_1 \end{cases}$$

Defining

$$u(x) := \begin{cases} \phi(x) & \text{if } x \in \Omega_o \\ \\ u_1(x) & \text{if } x \in \Omega_1, \end{cases}$$

it is clear that we find a function satisfying the statement of the theorem. □

Corollary 6.1. *assume that the set K is bounded and connected and that $\zeta_0 \in K$. Then there exists $u \in u_{\zeta_0} + W_0^{m,\infty}(\Omega; \mathbb{R}^N)$ such that u is a solution of (I_1).*

Proof. Set $\phi(x) = u_{\zeta_0}(x)$, and $E = \partial K$. Since K is bounded and connected it follows thatt E is compact and that $K \subset$ co∂K. Thus,

$$K \subseteq E \cup \text{intco}E$$

and so $\zeta_0 \in E \cup$ intco$E$. The existence of a solution to problem (I_1) follows immediately from Theorem (6.2). □

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