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Some applications of Gröbner bases in homological algebra

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Abstract. In this paper we make some computations in homological algebra using Gröbner bases for modules over polynomials rings with coefficients in a Noetherian commutative ring. In particular, we show easy procedures for computing the Ext and Tor modules.
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1. Introduction

Many concepts in algebra are easy to formulate but in concrete situations it is not easy to make computations involving them. For example, if M is a submodule of the free module A^m, where $A = R[x_1, \ldots, x_n]$ and R is a Noetherian commutative ring, a presentation of M is defined by the quotient A^s/K, where s is the number of generators of M and K is the kernel of the canonical homomorphism $A^s \longrightarrow M$ that sends the canonical vectors of A^s into the generators of M. However, for concrete examples the computation of K is not an easy task. For instance, if $M = < \boldsymbol{f}_1, \boldsymbol{f}_2 > \subseteq (\mathbb{Z}_{10}[x, y])^2$, where $\boldsymbol{f}_1 = (3x^2y + 3x, xy - 2y)$ and $\boldsymbol{f}_2 = (7xy^2 + y, y^2 - 4x)$ then a direct computation of K without a Computer Algebra System, or at least without an algorithmic procedure, is a very extensive an complicated exercise. Perhaps the most important algorithmic tools for solving this kind of problems are the Gröbner bases introduced in the sixties by Bruno Buchberger (see [3]). Gröbner bases play a key role in computational commutative algebra and let us to solve many problems that are practically impossible to attack with traditional theoretical methods.

Gröbner bases have been studied intensively in the last years, and there are a lot of interesting applications in many branches of mathematics such as in homological algebra, commutative algebra, algebraic geometry, differential algebra, graph theory, etc. Gröbner bases have been also used in applied sciences as statistics ([4]), robotic ([6]), linear control systems ([5],[14]), etc. There are excellent monographs that study the theory and applications of Gröbner bases. In particular, we can mention the recent works of Lorenzo Robbiano and Martin Kreuzer (see [11],[12] and also [9]), the monograph of Pfister and Greuel ([8]) that includes the package SINGULAR ([10]), the monographs of Cox *et al* with applications in Algebraic Geometry ([6], [7]), the classical book of Becker and Weispfenning ([2]), and the textbook of Adams and Loustaunau ([1]).

In this paper we present some applications of Gröbner bases of modules in homological algebra. Thus, if R is a Noetherian commutative ring, $A = R[x_1, \ldots, x_n]$, and A^m is the free module of vector columns of length $m \geq 1$ with entries in $R[x_1, \ldots, x_n]$, we will compute presentations of $Hom_A(M, N)$, $M \otimes_A N$, $Ext^r_A(M, N)$ and $Tor^A_r(M, N)$, where M is a given submodule of A^m and N is a given submodule of A^l, with $m, l \geq 1$ and $r \geq 0$. The technique we will use is very simple, we compute presentations of submodules of A^m using syzygies and Gröbner bases, and with this, we compute free resolutions and the correspondent modules of homology.

The theory and methods of Gröbner bases for modules are well known. For example, in [1] is presented the theory of Gröbner bases for ideals of A and for submodules of $(K[x_1, \ldots, x_n])^m$, where K is a field (see also the Chapter 2 of [8], and Chapters 2 and 3 of [11]). In [13] and [15] the theory was extended for submodules of A^m, in particular, in [13] was presented and proved the algorithm of Buchbeger for computing Gröbner bases of submodules of A^m. In the present paper we will use the Buchberger's algorithm of [13] for computing all Gröbner bases needed in the examples that illustrate our results and procedures below. Moreover, we will use the usual terminology about monomial orders on A and A^m (see [1] and [8]), in particular, we will use the POTREV order on monomials of A^m defined as in [13], i.e., given a monomial order $>$ on $Mon(A)$ (monomials of A), we define the following natural order on $Mon(A^m)$ (monomials of A^m).

Definition 1. *Let $\boldsymbol{X} = X\boldsymbol{e}_i$ and $\boldsymbol{Y} = Y\boldsymbol{e}_j \in Mon(A^m)$, where $X, Y \in Mon(A)$ and $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$ is the canonical basis of A^m. The POTREV order is defined by*

$$\boldsymbol{X} > \boldsymbol{Y} \Longleftrightarrow \begin{cases} i < j \\ or \\ i = j \text{ and } \quad X > Y. \end{cases}$$

Note that for this monomial order

$$\boldsymbol{e}_1 > \boldsymbol{e}_2 > \cdots > \boldsymbol{e}_m.$$

If $\boldsymbol{f} \neq \boldsymbol{0}$ is a vector of A^m, then we may write $\boldsymbol{f}$ as a sum of terms in the following way

$$\boldsymbol{f} = c_1\boldsymbol{X}_1 + \cdots + c_t\boldsymbol{X}_t,$$

where $c_1, \ldots, c_t \in R - 0$ and $\boldsymbol{X}_1 > \boldsymbol{X}_2 > \cdots > \boldsymbol{X}_t$ are monomials of $Mon(A^m)$.

Definition 2. *With the above notation, we say that*

(i) $lt(\boldsymbol{f}) = c_1\boldsymbol{X}_1$ *is the leading term of* $\boldsymbol{f}$.
(ii) $lc(\boldsymbol{f}) = c_1$ *is the leading coefficient of* $\boldsymbol{f}$.
(iii) $lm(\boldsymbol{f}) = \boldsymbol{X}_1$ *is the leading monomial of* $\boldsymbol{f}$.

With the above terminology we recall the definition of Gröbner basis for submodules of A^m (see [13]): Let $G \neq 0$ be a non empty subset of A^m, then

$$Lt(G) = < lt(\boldsymbol{g}) | \boldsymbol{g} \in G >$$

is the submodule of A^m generated for the leading terms of vectors of G. Let $M \neq 0$ be a submodule of A^m and let $G \neq 0$ be a non empty finite subset of M, we say that G is a *Gröbner basis* for M if $Lt(M) = Lt(G)$. Taking $m = 1$ we get the definition of Gröbner basis for ideals of A. An effective method for computing Gröbner bases of ideals and submodules over commutative polynomials rings is the the Buchberger's algorithm that can be found in [1] or [13]. For more details about the general theory of Gröbner bases of ideals and modules see [1], [8] and [13].

The paper in divided in four sections. The second section is dedicated to compute syzygies of modules, the presentation of a given module and the kernel and the image of a homomorphism between modules. In the third section we compute the modules $Hom_A(M, N)$ and $M \otimes_A N$, and also we will compute a free resolution of a given submodule M of A^m. The last section is dedicated to compute presentations for $Ext^r_A(M, N)$ and $Tor^A_r(M, N)$. All computations will be illustrated with examples. We remark that in the literature is difficult to find such illustrative examples.

2. Elementary applications of Gröbnes bases

In this section we will list the most basic applications of Gröbner bases in module theory that we will use later. We will compute syzygies of modules, the presentation of a given module, the kernel and the image of a homomorphism between modules. All of these computations are analogues of those for ideals of $R[x_1, \ldots, x_n]$ or submodules of $K[x_1, \ldots, x_n]^m$, and the proofs have been adapted (see Chapters 3 and 4 in [1], and Chapter 2 of [8]).

2.1. **Syzygy of a module.** We start computing the syzygy of a submodule $M =< \boldsymbol{f}_1, \dots, \boldsymbol{f}_s >$ of A^m. For this we consider the matrix $F = [\boldsymbol{f}_1 \cdots \boldsymbol{f}_s]$ and we recall that $Syz(F)$ consists of vectors $\boldsymbol{h} = (h_1, \dots, h_s) \in A^s$ such that

$$h_1\boldsymbol{f}_1 + \cdots + h_s\boldsymbol{f}_s = \mathbf{0},$$

i.e., $F\boldsymbol{h} = \mathbf{0}$. We note that $Syz(F)$ is a submodule of A^s and we also define $Syz(M) := Syz(F)$.

In order to compute $Syz(M)$ we need the following preliminary result (see Lemma 2.8.2 in [8]).

Theorem 3. *Let $F = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_s\}$ be a set of non-zero vectors of A^m and G a Gröbner basis for $M =< F >$ with respect to POTREV order on $Mon(A^m)$. Then, for any $k = 0, \dots, m-1$, $G \cap (\bigoplus_{i=k+1}^m A\boldsymbol{e}_i)$ is a Gröbner basis for $M \cap (\bigoplus_{i=k+1}^m A\boldsymbol{e}_i)$.*

The key for computing syzygies is the following theorem.

Theorem 4. *Let $F = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_s\}$ be a set of non-zero vectors of A^m. Consider the canonical embedding*

$$A^m \longrightarrow A^{m+s}$$

and the canonical projection

$$\pi : A^{m+s} \longrightarrow A^s.$$

Let $G = \{\boldsymbol{g}_1, \dots, \boldsymbol{g}_t\}$ be a Gröbner basis for $< \boldsymbol{f}_1 + \boldsymbol{e}_{m+1}, \dots, \boldsymbol{f}_s + \boldsymbol{e}_{m+s} >$ with respect to the POTREV order on $Mon(A^{m+s})$. If $G \cap (\bigoplus_{i=m+1}^{m+s} A\boldsymbol{e}_i) = \{\boldsymbol{g}_1, \dots, \boldsymbol{g}_l\}$, then $Syz(F) =< \pi(\boldsymbol{g}_1), \dots, \pi(\boldsymbol{g}_l) >$.

Proof. If K is a field the proof of the Lemma 2.5.3 of [8] for $A = K[x_1, \dots, x_n]$ applies in our general situation. We will adapt this proof using a matrix notation. Let $M =< \boldsymbol{f}_1 + \boldsymbol{e}_{m+1}, \dots, \boldsymbol{f}_s + \boldsymbol{e}_{m+s} >$, by the previous theorem $\{\boldsymbol{g}_1, \dots, \boldsymbol{g}_l\}$ is a Gröbner basis for $M \cap (\bigoplus_{i=m+1}^{m+s} A\mathbf{e}_i)$. For $1 \leq v \leq l$, let

$$\boldsymbol{g}_v = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g_{m+1,v} \\ \vdots \\ g_{m+s,v} \end{bmatrix}.$$

We know that during the process of computing a Gröbner basis we can express each $\boldsymbol{g}_i$ as a combination of generators of M (see [13]), thus there exist elements $h_1, \ldots, h_s \in A$ such that

$$\boldsymbol{g}_v = h_1 \begin{bmatrix} f_{11} \\ \vdots \\ f_{m1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + h_s \begin{bmatrix} f_{1s} \\ \vdots \\ f_{ms} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

where by the canonical embedding

$$\boldsymbol{f}_i = \begin{bmatrix} f_{1i} \\ \vdots \\ f_{mi} \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}.$$

then $h_i = g_{m+i,v}$ for each $1 \leq i \leq s$ and also

$$h_1 \begin{bmatrix} f_{11} \\ \vdots \\ f_{m1} \end{bmatrix} + \cdots + h_s \begin{bmatrix} f_{1s} \\ \vdots \\ f_{ms} \end{bmatrix} = 0,$$

i.e., $g_{m+1,v}\boldsymbol{f}_1 + \cdots + g_{m+s,v}\boldsymbol{f}_s = 0$, and hence, $\pi(\boldsymbol{g}_v) \in Syz(F)$ for each $1 \leq v \leq l$. This proves that $< \pi(\boldsymbol{g}_1), \ldots, \pi(\boldsymbol{g}_l) > \subseteq Syz(F)$.

Conversely, let $\boldsymbol{h} = (h_1, \ldots, h_s) \in Syz(F)$, then $h_1\boldsymbol{f}_1 + \cdots + h_s\boldsymbol{f}_s = \mathbf{0}$ and hence

$$h_1\begin{bmatrix}f_{11}\\ \vdots\\ f_{m1}\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix}+\cdots+h_s\begin{bmatrix}f_{1s}\\ \vdots\\ f_{ms}\\ 0\\ 0\\ \vdots\\ 1\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\\ h_1\\ h_2\\ \vdots\\ h_s\end{bmatrix}\in M\cap(\bigoplus_{i=m+1}^{m+s}A\mathbf{e}_i).$$

Since $\{\boldsymbol{g}_1,\ldots,\boldsymbol{g}_l\}$ is a Gröbner basis for $M\cap(\bigoplus_{i=m+1}^{m+s}A\mathbf{e}_i)$, there exist $p_1,\ldots,p_l\in A$ such that

$$\begin{bmatrix}0\\ \vdots\\ 0\\ h_1\\ \vdots\\ h_s\end{bmatrix}=p_1\begin{bmatrix}0\\ \vdots\\ 0\\ g_{m+1,1}\\ \vdots\\ g_{m+s,1}\end{bmatrix}+\cdots+p_l\begin{bmatrix}0\\ \vdots\\ 0\\ g_{m+1,l}\\ \vdots\\ g_{m+s,l}\end{bmatrix},$$

hence $\boldsymbol{h}\in<\pi(\boldsymbol{g}_1),\ldots,\pi(\boldsymbol{g}_l)>$. This complete the proof. □

The following example illustrates the elimination method described in the previous theorem.

Example 5. In [13] we computed a Gröbner basis for $M=<\boldsymbol{f}_1,\boldsymbol{f}_2>$ with respect to POTREV order on $Mon(A^2)$, where $\boldsymbol{f}_1=3ix^2y\boldsymbol{e}_1+(1+i)z\boldsymbol{e}_1+2ixy^2\boldsymbol{e}_2+5z\boldsymbol{e}_2$, $\boldsymbol{f}_2=(2+i)x^2z\boldsymbol{e}_1+y\boldsymbol{e}_1+3xy^2\boldsymbol{e}_2+y^2\boldsymbol{e}_2+4iz\boldsymbol{e}_2$ and $A=\mathbb{Z}[i][x,y,z]$. On $Mon(A)$ we used the order deglex with $x>y>z$. The Gröbner basis we computed is $G=\{\boldsymbol{f}_1,\boldsymbol{f}_2,\boldsymbol{f}_3,\boldsymbol{f}_4\}$, where $\boldsymbol{f}_3=3y^2\boldsymbol{e}_1+(-3+i)z^2\boldsymbol{e}_1+9xy^3\boldsymbol{e}_2+(-4-2i)xy^2z\boldsymbol{e}_2+3y^3\boldsymbol{e}_2+12iyz\boldsymbol{e}_2+(-5+10i)z^2\boldsymbol{e}_2$ and $\boldsymbol{f}_4=9x^3y^3\boldsymbol{e}_2+(-4-2i)x^3y^2z\boldsymbol{e}_2+3x^2y^3\boldsymbol{e}_2+12ix^2yz\boldsymbol{e}_2+(-5+10i)x^2z^2\boldsymbol{e}_2-2xy^3\boldsymbol{e}_2+(3-3i)xy^2z\boldsymbol{e}_2+(1-i)y^2z\boldsymbol{e}_2+5iyz\boldsymbol{e}_2+(4+4i)z^2\boldsymbol{e}_2$. Using again the algorithm of Buchberger presented in the Theorem 23 of [13] we compute a Gröbner basis G' for $F'=\{\boldsymbol{f}_1+\boldsymbol{e}_3,\boldsymbol{f}_2+\boldsymbol{e}_4\}$,

$$G'=\{\boldsymbol{g}'_1,\boldsymbol{g}'_2,\boldsymbol{g}'_3,\boldsymbol{g}'_4\},$$

where $\boldsymbol{g}'_1=\boldsymbol{f}_1+\boldsymbol{e}_3$, $\boldsymbol{g}'_2=\boldsymbol{f}_2+\boldsymbol{e}_4$, $\boldsymbol{g}'_3=\boldsymbol{f}_3+(-1+2i)z\boldsymbol{e}_3+3y\boldsymbol{e}_4$ and $\boldsymbol{g}'_4=\boldsymbol{f}_4+(-1+2i)x^2z\boldsymbol{e}_3+iy\boldsymbol{e}_3+3x^2y\boldsymbol{e}_4+(1-i)z\boldsymbol{e}_4$. We observe that $G'\cap(Ae_3\oplus Ae_4)=\emptyset$, and hence $Syz(\boldsymbol{f}_1,\boldsymbol{f}_2)=0$.

2.2. **Presentation of a module.** Let $M =< \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s >$ be a submodule of A^m, there exists a natural surjective homomorphism $\pi_M : A^s \longrightarrow M$ defined by $\pi_M(\boldsymbol{e}_i) = \boldsymbol{f}_i$, where $\{\boldsymbol{e}_i\}_{1\leq i\leq s}$ is the canonical basis of A^s. If $K_M = \ker(\pi_M)$, then we have the isomorphism $\overline{\pi_M} : A^s/K_M \cong M$, defined by $\overline{\pi_M}(\overline{\boldsymbol{e}_i}) = \boldsymbol{f}_i$, where $\overline{\boldsymbol{e}_i} = \boldsymbol{e}_i + K_M$. We note that $K_M =< h_1, \ldots, h_{s_1} >$ is also a finitely generated module and we have the exact sequence

$$A^{s_1} \xrightarrow{\delta_M} A^s \xrightarrow{\pi_M} M \longrightarrow 0, \tag{2.1}$$

with $\delta_M = i_M \circ \pi'_M$, where i_M is the inclusion of K_M in A^s and π'_M is the natural surjective homomorphism from A^{s_1} to K_M. We recall that the quotient module A^s/K_M, or equivalently, the exact sequence (2.1), is a presentation of M. We observe that $K_M = Syz(M) = Syz(F)$, where $F = [\boldsymbol{f}_1 \cdots \boldsymbol{f}_s]$, and consequently, the Theorem 4 gives a method for computing a presentation of a module. On the other hand, let Δ_M be the matrix of δ_M in the canonical bases of A^{s_1} and A^s, then the columns of Δ_M are the generators of $Syz(F)$ since $Im(\delta_M) = \ker(\pi_M)$. We will also say that Δ_M is a *matrix presentation* of M.

The next example will be used often in this paper.

Example 6. Let $M =< \boldsymbol{f}_1, \boldsymbol{f}_2 >\subseteq (\mathbb{Z}_{10}[x,y])^2$ and $N =< \boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3 >\subseteq (\mathbb{Z}_{10}[x,y])^2$, where $\boldsymbol{f}_1 = (3x^2y + 3x, xy - 2y), \boldsymbol{f}_2 = (7xy^2 + y, y^2 - 4x), \boldsymbol{g}_1 = (0, x)$ and $\boldsymbol{g}_2 = (y, x)$ and $\boldsymbol{g}_3 = (2x, x)$, then applying the Theorem 4 with order deglex on $Mon(\mathbb{Z}[x,y])$ and $x > y$, we get presentations for M and N,

$$M \cong A^2/Syz(M),\ N \cong A^3/Syz(N),$$

where

$$Syz(M) =< (5y, 5x) >,\ Syz(N) =< (5, 0, 5), (2x + 9y, 8x, y) >.$$

Presentations of quotient modules could be also computed. In fact, let $N \subseteq M$ be submodules of A^m, $M =< \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s >$, $N =< \boldsymbol{g}_1, \ldots, \boldsymbol{g}_t >$, $M/N =< \overline{\boldsymbol{f}_1}, \ldots, \overline{\boldsymbol{f}_s} >$, then we have a canonical surjective homomorphism $\delta : A^s \longrightarrow M/N$ such that a presentation of M/N is given by $M/N \cong A^s/Syz(M/N)$. But $Syz(M/N)$ can be computed in the following way. $\boldsymbol{h} = (h_1, \ldots, h_s) \in Syz(M/N)$ if and only if $h_1\boldsymbol{f}_1 + \cdots + h_s\boldsymbol{f}_s \in< \boldsymbol{g}_1, \ldots, \boldsymbol{g}_t >$ if and only if there exist $h_{s+1}, \ldots, h_{s+t} \in A$ such that $h_1\boldsymbol{f}_1 + \cdots + h_s\boldsymbol{f}_s + h_{s+1}\boldsymbol{g}_1 + \cdots + h_{s+t}\boldsymbol{g}_t = \boldsymbol{0}$ if and only if $(h_1, \ldots, h_s, h_{s+1}, \ldots, h_{s+t}) \in Syz(H)$, where

$$H = [\boldsymbol{f}_1\ \cdots\ \boldsymbol{f}_s\ \boldsymbol{g}_1\ \cdots\ \boldsymbol{g}_t].$$

Thus we have the following well know result.

Theorem 7. *With the notation above, a presentation of M/N is given by $A^s/Syz(M/N)$, where a set of generators of $Syz(M/N)$ are the first s coordinates of generators of $Syz(H)$.*

2.3. **Kernel and image of a homomorphism.** Let $M \subseteq A^m$ and $N \subseteq A^l$ be modules, $M =< \boldsymbol{f}_1, \dots, \boldsymbol{f}_s >$, $N =< \boldsymbol{g}_1, \dots, \boldsymbol{g}_t >$, and let $\phi : M \longrightarrow N$ be a homomorphism. Then, there exists a matrix $\Phi = [\phi_{ji}]$ of size $t \times s$ with entries in A defined by

$$\phi(\boldsymbol{f}_i) = \phi_{1i}\boldsymbol{g}_1 + \cdots + \phi_{ti}\boldsymbol{g}_t,$$

for each $1 \le i \le s$. We compute now a system of generators and a presentation for $\ker \phi$ and $Im(\phi)$. We assume that the homomorphism ϕ is well-defined and it is given by the matrix Φ. Using the notation of the previous subsection, let $A^s/Syz(M)$ and $A^t/Syz(N)$ be presentations of M and N. We consider the canonical isomorphisms

$$\overline{\pi_M} : A^s/Syz(M) \longrightarrow M, \ \overline{\pi_N} : A^t/Syz(N) \longrightarrow N$$

defined by $\overline{\pi_M}(\overline{\boldsymbol{e}_i}) = \boldsymbol{f}_i$, $1 \le i \le s$, $\overline{\pi_N}(\overline{\boldsymbol{e}'_j}) = \boldsymbol{g}_j$, $1 \le j \le t$, where $\{\boldsymbol{e}_i\}_{1 \le i \le s}$ is the canonical basis of A^s and $\{\boldsymbol{e}'_j\}_{1 \le j \le t}$ is the canonical basis of A^t. Then, we have the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow & & \downarrow \\ A^s/Syz(M) & \xrightarrow{\overline{\phi}} & A^t/Syz(N) \end{array} \tag{2.2}$$

where the vertical arrows are the isomorphisms $(\overline{\pi_M})^{-1}$ and $(\overline{\pi_N})^{-1}$. Hence, $\overline{\phi}(\overline{\boldsymbol{e}_i}) = (\overline{\pi_N})^{-1} \circ \phi \circ \overline{\pi_M}(\overline{\boldsymbol{e}_i}) = \phi_{1i}\overline{\boldsymbol{e}'_1} \cdots + \phi_{ti}\overline{\boldsymbol{e}'_t}$, for each $1 \le i \le s$. We observe that the matrix of $\overline{\phi}$ coincides with the matrix of ϕ, $\ker(\phi) \cong \ker(\overline{\phi})$ and $Im(\phi) \cong Im(\overline{\phi})$.

Let $h_1\boldsymbol{f}_1 + \cdots + h_s\boldsymbol{f}_s \in \ker(\phi)$, then $(\overline{\pi_N})^{-1}(\phi(h_1\boldsymbol{f}_1 + \cdots + h_s\boldsymbol{f}_s)) = \overline{\boldsymbol{0}} = \overline{\phi}((\overline{\pi_M})^{-1}(h_1\boldsymbol{f}_1 + \cdots + h_s\boldsymbol{f}_s)) = \overline{\phi}(h_1\overline{\boldsymbol{e}_1} + \cdots + h_s\overline{\boldsymbol{e}_s}) = h_1\overline{\phi}(\overline{\boldsymbol{e}_1}) + \cdots + h_s\overline{\phi}(\overline{\boldsymbol{e}_s}) = h_1(\phi_{11}\overline{\boldsymbol{e}'_1} + \cdots + \phi_{t1}\overline{\boldsymbol{e}'_t}) + \cdots + h_s(\phi_{1s}\overline{\boldsymbol{e}'_1} + \cdots + \phi_{ts}\overline{\boldsymbol{e}'_t}) = (h_1\phi_{11} + \cdots + h_s\phi_{1s})\overline{\boldsymbol{e}'_1} + \cdots + (h_1\phi_{t1} + \cdots + h_s\phi_{ts})\overline{\boldsymbol{e}'_t}$. This implies that $(h_1\phi_{11} + \cdots + h_s\phi_{1s})\boldsymbol{e}'_1 + \cdots + (h_1\phi_{t1} + \cdots + h_s\phi_{ts})\boldsymbol{e}'_t \in Syz(N)$. We assume that we have computed a system of generators for $Syz(N) =< \boldsymbol{u}_1, \dots, \boldsymbol{u}_{t_1} > \subseteq A^t$ (Theorem 4). Hence, there exist $h_{s+1}, \dots, h_{s+t_1} \in A$ such that

$$h_1 \begin{bmatrix} \phi_{11} \\ \vdots \\ \phi_{t1} \end{bmatrix} + \cdots + h_s \begin{bmatrix} \phi_{1s} \\ \vdots \\ \phi_{ts} \end{bmatrix} + h_{s+1}\boldsymbol{u}_1 + \cdots + h_{s+t_1}\boldsymbol{u}_{t_1} = \boldsymbol{0}.$$

From these computations we also can conclude that

$$h_1\boldsymbol{f}_1 + \cdots + h_s\boldsymbol{f}_s \in \ker(\phi) \Leftrightarrow \overline{(h_1, \ldots, h_s)} \in \ker(\overline{\phi}).$$

Thus, we have proved the following theorem.

Theorem 8. *With the notation above, let*

$$H = [\Phi_1 \quad \cdots \quad \Phi_s \quad \boldsymbol{u}_1 \quad \cdots \quad \boldsymbol{u}_{t_1}],$$

where Φ_i is the i^{th} column of the matrix Φ, $1 \leq i \leq s$. Then,

$$(h_1, \ldots, h_s, h_{s+1}, \ldots, h_{s+t_1}) \in Syz(H) \Leftrightarrow h_1\boldsymbol{f}_1 + \cdots + h_s\boldsymbol{f}_s \in \ker(\phi).$$

Thus, if $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_v\} \subset A^{s+t_1}$ is a system of generators of $Syz(H)$, let $\boldsymbol{z}'_k \in A^s$ be the vector obtained from $\boldsymbol{z}_k$ when omitting the last t_1 components, $1 \leq k \leq v$, then $\{\overline{\boldsymbol{z}'_1}, \ldots, \overline{\boldsymbol{z}'_v}\}$ is a system of generators for $\ker(\overline{\phi})$. Moreover, if

$$\boldsymbol{z}'_1 = (h_{11}, \ldots, h_{1s}), \ldots, \boldsymbol{z}'_v = (h_{v1}, \ldots, h_{vs}),$$

then $\{h_{11}\boldsymbol{f}_1 + \cdots + h_{1s}\boldsymbol{f}_s, \ldots, h_{v1}\boldsymbol{f}_1 + \cdots + h_{vs}\boldsymbol{f}_s\}$ is a system of generators for $\ker(\phi)$.

A presentation of $\ker(\phi)$ is given in the following way.

Corollary 9. *With the notation of this section, a presentation of $\ker(\phi)$ is given by A^v/K, where*

$$K = Syz(\ker(\phi)) = Syz\,[h_{11}\boldsymbol{f}_1 + \cdots + h_{1s}\boldsymbol{f}_s \quad \cdots \quad h_{v1}\boldsymbol{f}_1 + \cdots + h_{vs}\boldsymbol{f}_s].$$

Example 10. In the *Step* 6 of Section 3.1 we will prove that the function ϕ defined by

$$\begin{aligned} M &\xrightarrow{\phi} N \\ \boldsymbol{f}_1 &\mapsto x\boldsymbol{g}_3 \\ \boldsymbol{f}_2 &\mapsto y\boldsymbol{g}_1 \end{aligned}$$

is an homomorphism, where M and N are as in the Example 6. Now we will calculate a system of generators for $\ker(\phi)$. We note that the matrix of ϕ is

$$\Phi = \begin{bmatrix} 0 & y \\ 0 & 0 \\ x & 0 \end{bmatrix},$$

thus, we must compute $Syz(H)$ where

$$H = \begin{bmatrix} 0 & y & 5 & 2x+9y \\ 0 & 0 & 0 & 8x \\ x & 0 & 5 & y \end{bmatrix}.$$

For this we apply the Theorem 4 and we get

$$Syz(H) =< (0,0,2,0), (0,0,-y,5) >.$$

By the Theorem 8, we omit the last two components of $(0,0,2,0)$, $(0,0,-y,5)$ and we get that $\ker(\phi) = 0$.

An explicit presentation for $\ker(\overline{\phi})$ could be also given. We assume that we have computed a system of generators for $Syz(M) =< \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{s_1} >\subseteq A^s$. We know that a presentation of $\ker(\overline{\phi})$ is given by $\ker(\overline{\phi}) \cong A^v/K'$, where $K' = Syz(\ker(\overline{\phi})) = Syz(< \overline{\boldsymbol{z}'_1}, \ldots, \overline{\boldsymbol{z}'_v} >)$. But, $(l_1, \ldots, l_v) \in Syz(< \overline{\boldsymbol{z}'_1}, \ldots, \overline{\boldsymbol{z}'_v} >)$ if and only if there exist $l_{v+1}, \ldots, l_{v+s_1} \in A$ such that $l_1\boldsymbol{z}'_1 + \cdots + l_v\boldsymbol{z}'_v + l_{v+1}\boldsymbol{w}_1 + \cdots + l_{v+s_1}\boldsymbol{w}_{s_1} = \mathbf{0}$. Thus, we have proved the following corollary.

Corollary 11. *With the notation above, let*

$$L = [\boldsymbol{z}'_1 \quad \cdots \quad \boldsymbol{z}'_v \quad \boldsymbol{w}_1 \quad \cdots \quad \boldsymbol{w}_{s_1}],$$

if $\{\boldsymbol{l}_1, \ldots, \boldsymbol{l}_q\} \subset A^{v+s_1}$ *is a system of generators of* $Syz(L)$, *let* $\boldsymbol{l}'_k \in A^v$ *be the vector obtained from* $\boldsymbol{l}_k$ *when omitting the last* s_1 *components,* $1 \leq k \leq q$, *then* $\{\boldsymbol{l}'_1, \ldots, \boldsymbol{l}'_q\}$ *is a system of generators for* K', *and hence, a presentation of* $\ker(\overline{\phi})$ *is given by* A^v/K'.

We consider now the image of homomorphism $\phi : M \longrightarrow N$ in (2.2). Then the following result is clear from the above discussion.

Corollary 12. *A system of generators for* $Im(\phi)$ *is given by*

$$Im(\phi) =< \phi_{11}\mathbf{g}_1 + \cdots + \phi_{t1}\mathbf{g}_t, \ldots, \phi_{1s}\mathbf{g}_1 + \cdots + \phi_{ts}\mathbf{g}_t >.$$

A presentation of $Im(\phi)$ *is* A^s/I, *where*

$$I = Syz\,[\phi_{11}\mathbf{g}_1 + \cdots + \phi_{t1}\mathbf{g}_t \quad \ldots \quad \phi_{1s}\mathbf{g}_1 + \cdots + \phi_{ts}\mathbf{g}_t].$$

Example 13. Let ϕ be as in the Example 10, then by the previous corollary, $Im(\phi) =< (2x^2, x^2), (0, xy) >$. A presentation of $Im(\phi)$ is given by $Syz(I)$, where

$$I = \begin{bmatrix} 2x^2 & 0 \\ x^2 & xy \end{bmatrix},$$

so we apply again the Theorem 4 and we get $Syz(I) =< (5y, 5x) >$. Thus, $Im(\phi) \cong A^2/ < (5y, 5x) >$. This conclusion of course coincides with the results of the Examples 6 and 10 since $Im(\phi) \cong M$.

We conclude this section showing an explicit presentation of $Im(\overline{\phi})$. We know that $Im(\overline{\phi}) =< \phi_{11}\overline{\boldsymbol{e}'_1} + \cdots + \phi_{t1}\overline{\boldsymbol{e}'_t}, \ldots, \phi_{1s}\overline{\boldsymbol{e}'_1} + \cdots + \phi_{ts}\overline{\boldsymbol{e}'_t} >$, thus a presentation of $Im(\overline{\phi})$ is given by $Im(\overline{\phi}) \cong A^s/Syz(Im(\overline{\phi}))$. Let $(h_1, \ldots, h_s) \in Syz(Im(\overline{\phi}))$, then there exist $h_{s+1}, \ldots, h_{s+t_1} \in A$ such that

$$h_1 \begin{bmatrix} \phi_{11} \\ \vdots \\ \phi_{t1} \end{bmatrix} + \cdots + h_s \begin{bmatrix} \phi_{1s} \\ \vdots \\ \phi_{ts} \end{bmatrix} + h_{s+1}\boldsymbol{u}_1 + \cdots + h_{s+t_1}\boldsymbol{u}_{t_1} = \boldsymbol{0}.$$

Thus, we have proved the following corollary.

Corollary 14. *Let H be the matrix in the Theorem 8. If $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_v\} \subset A^{s+t_1}$ is a system of generators of $Syz(H)$, let $\boldsymbol{z}'_k \in A^s$ be the vector obtained from $\boldsymbol{z}_k$ when omitting the last t_1 components, $1 \leq k \leq v$, then $\{\boldsymbol{z}'_1, \ldots, \boldsymbol{z}'_v\}$ is a system of generators for $Syz(Im(\overline{\phi}))$ and $A^s/Syz(Im(\overline{\phi}))$ is a presentation of $Im(\overline{\phi})$.*

3. Computing Hom, ⊗ and free resolutions

The main goal of the paper is to compute the modules Ext and Tor, in this section we will compute the particular cases $Ext^0_A(M,N) = Hom_A(M,N)$ and $Tor^A_0(M,N) = M \otimes_A N$. We also show how to compute free resolutions.

3.1. Computation of Hom.

In this subsection we show a procedure for computing $Hom_A(M,N)$, where M is a submodule of A^m and N is a submodule of A^l. By computing $Hom_A(M,N)$ we mean to find a presentation of $Hom_A(M,N)$ and to find an specific set of generators for $Hom_A(M,N)$. For fields, i.e., when $A = K[x_1, \ldots, x_n]$, where K is a field, the computation of $Hom_A(M,N)$ can be found in [1], [8] and [11]. The constructions there apply to our more general situation when the ring of coefficients of A is a Noetherian commutative ring R, however we will add the explicit definition of some homomorphisms and we will prove the commutativity of some diagrams omitted in the literature.

We will illustrate the theory and steps of the procedure through the following particular example: $M =< \boldsymbol{f}_1, \boldsymbol{f}_2 >\subseteq (\mathbb{Z}_{10}[x,y])^2$ and $N =< \boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3 >\subseteq (\mathbb{Z}_{10}[x,y])^2$, where $\boldsymbol{f}_1 = (3x^2y + 3x, xy - 2y), \boldsymbol{f}_2 = (7xy^2 + y, y^2 - 4x), \boldsymbol{g}_1 = (0,x)$ and $\boldsymbol{g}_2 = (y,x)$ and $\boldsymbol{g}_3 = (2x,x)$ (see the Example 6).

Let M and N be submodules of A^m and A^l, respectively, then M and N are finitely generated modules, $M =< \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s >$, $N =< \boldsymbol{g}_1, \ldots, \boldsymbol{g}_t >$. In our concrete example, $A = \mathbb{Z}_{10}[x,y]$, and we choose the POTREV order for the monomials of A^2. The order in $Mon(A)$ is deglex with $x > y$. So, in this illustrative example, $R = \mathbb{Z}_{10}$, $A = \mathbb{Z}_{10}[x, y]$, $m = 2, s = 2, l = 2, t = 3$. We divide the procedure in some steps.

Step 1. Presentations of M and N. In order to compute a presentation of $Hom_A(M, N)$ we first compute presentations of M and N as we saw in the second section:

$$M \cong A^s/K_M, N \cong A^t/K_N,$$

where K_M and K_N are the kernels of natural homomorphisms $\pi_M : A^s \longrightarrow M$ and $\pi_N : A^t \longrightarrow N$ defined by $\pi_M(\boldsymbol{e}_i) = \boldsymbol{f}_i$, $\pi_N(\boldsymbol{e}'_j) = \boldsymbol{g}_j$, $1 \leq i \leq s$, $1 \leq j \leq t$ ($\{\boldsymbol{e}_i\}_{1\leq i\leq s}$ is the canonical basis of A^m and $\{\boldsymbol{e}'_j\}_{1\leq j\leq t}$ is the canonical basis of A^l). Then, $Hom_A(M, N) \cong Hom_A(A^s/K_M, A^t/K_N)$, and we can compute a presentation and a system of generators of $Hom_A(A^s/K_M, A^t/K_N)$ instead of $Hom_A(M, N)$. However, in the last step of the procedure we will use the system of generators of $Hom_A(A^s/K_M, A^t/K_N)$ for giving an explicit system of generators of $Hom_A(M, N)$.

We recall that K_M and K_N are computed by the syzygies of the matrices

$$F_M = [\boldsymbol{f}_1 \quad \cdots \quad \boldsymbol{f}_s], F_N = [\boldsymbol{g}_1 \quad \cdots \quad \boldsymbol{g}_t],$$

i.e., $K_M = Syz(F_M) = Syz(M), K_N = Syz(F_N) = Syz(N)$. In our example,

$$M \cong A^2/Syz(M), N \cong A^3/Syz(N),$$

and by the Example 6,

$$Syz(M) =< (5y, 5x) >, Syz(N) =< (5, 0, 5), (2x + 9y, 8x, y) >.$$

Hence,

$$Hom_A(M, N) \cong$$
$$\cong Hom_A(A^2/ < (5y, 5x) >, A^3/ < (5, 0, 5), (2x + 9y, 8x, y) >).$$

Step 2. $Hom_A(A^s/Syz(M), A^t/Syz(N))$ as a kernel. Since R is a Noetherian ring, $Syz(M)$ and $Syz(N)$ are finitely generated A-modules, $Syz(M)$ is generated by s_1 elements and $Syz(N)$ is generated by t_1 elements. Thus, we also have surjective homomorphisms $\pi'_M : A^{s_1} \longrightarrow Syz(M)$ and $\pi'_N : A^{t_1} \longrightarrow Syz(N)$, and hence the following sequences are exact

$$A^{s_1} \xrightarrow{\delta_M} A^s \xrightarrow{j_M} A^s/Syz(M) \longrightarrow 0 \tag{3.1}$$

$$A^{t_1} \xrightarrow{\delta_N} A^t \xrightarrow{j_N} A^t/Syz(N) \longrightarrow 0 \tag{3.2}$$

where $\delta_M = i_M \circ \pi'_M$, $\delta_N = i_N \circ \pi'_N$, i_M, i_N denote inclusions, and j_M, j_N are natural homomorphisms. From (3.1) we get the exact sequence

$$0 \to Hom_A(A^s/Syz(M), A^t/Syz(N)) \xrightarrow{p} Hom_A(A^s, A^t/Syz(N)) \xrightarrow{d} Hom_A(A^{s_1}, A^t/Syz(N)),$$

where

$$d(\alpha) = \alpha \circ \delta_M \quad \text{for} \quad \alpha \in Hom_A(A^s, A^t/Syz(N))$$

and p is defined in the same way. Thus,

$$Hom_A(A^s/Syz(M), A^t/Syz(N)) \cong \ker(d). \tag{3.3}$$

Step 3. Computation of $Hom_A(A^s, A^t/Syz(N))$ *and* $Hom_A(A^{s_1}, A^t/Syz(N))$. According to (3.3), we must compute presentations of $Hom_A(A^s, A^t/Syz(N))$ and $Hom_A(A^{s_1}, A^t/Syz(N))$. Let Δ_M be the matriz presentation of M and Δ_N the matrix presentation of N.

In our example, $s_1 = 1$, $t_1 = 2$ and

$$\Delta_M = \begin{bmatrix} 5y \\ 5x \end{bmatrix}, \ \Delta_N = \begin{bmatrix} 5 & 2x+9y \\ 0 & 8x \\ 5 & y \end{bmatrix}.$$

Let $M_{t_1 s}(A)$ be the ring of matrices with t_1 rows and s columns with entries in A, in the same way we define $M_{ts}(A)$. Then, from (3.2) we get the diagram

$$\begin{array}{ccccc} Hom_A(A^s, A^{t_1}) & \xrightarrow{d_N} & Hom_A(A^s, A^t) & \xrightarrow{p_N} & Hom_A(A^s, A^t/Syz(N)) \\ \downarrow & & \downarrow & & \\ M_{t_1 s}(A) & & M_{ts}(A) & & \\ \downarrow & & \downarrow & & \\ A^{t_1 s} & \xrightarrow{d_N^*} & A^{ts} & & \end{array} \tag{3.4}$$

where d_N is the natural homomorphism induced by δ_N, i.e.,

$$d_N(\beta) = \delta_N \circ \beta \ \text{ for } \ \beta \in Hom_A(A^s, A^{t_1})$$

and p_N is surjective and defined in the same way. The first row is exact since A^s is projective. $A^{t_1 s}$ is the free module of vector columns of size $t_1 s$ obtained by concatenating the columns of matrices of $M_{t_1 s}(A)$. A^{ts} is defined in the same way, thus the vertical arrows are natural isomorphisms. The left vertical compose isomorphism is noted by Φ_{s,t_1} and the right compose isomorphism is noted by $\Phi_{s,t}$. d_N^* is induced by d_N and the vertical isomorphisms, $d_N^* = \Phi_{s,t} \circ d_N \circ \Phi_{s,t_1}^{-1}$. Then, the following diagram is commutative:

$$\begin{array}{ccc} Hom_A(A^s, A^{t_1}) & \xrightarrow{d_N} & Hom_A(A^s, A^t) \\ \Phi_{s,t_1}\downarrow & & \downarrow\Phi_{s,t} \\ A^{t_1 s} & \xrightarrow{d_N^*} & A^{ts} \end{array}$$

We can explicit how acts the homomorphism d_N^*, in fact, if

$$\boldsymbol{a} = (a_{11}, \ldots, a_{t_1 1}, \ldots, a_{1s}, \ldots, a_{t_1 s}) \in A^{t_1 s}$$

and $\Delta_N = [\Delta_{ij}]$ is the matrix of δ_N in the canonical bases of A^{t_1} and A^t, then

$$d_N^*(\boldsymbol{a}) = (\sum_{k=1}^{t_1} \Delta_{1k} a_{k1}, \ldots, \sum_{k=1}^{t_1} \Delta_{tk} a_{k1}, \ldots, \sum_{k=1}^{t_1} \Delta_{1k} a_{ks}, \ldots, \sum_{k=1}^{t_1} \Delta_{tk} a_{ks}) \tag{3.5}$$

Thus, in the canonical bases of $A^{t_1 s}$ and A^{ts} the matrix of d_N^* is

$$I_s \otimes \Delta_N$$

where I_s is the identical matrix of size $s\times$ and $\otimes$ means tensor product. Thus, $d_N^*(\boldsymbol{a}) = (I_s \otimes \Delta_N)\boldsymbol{a}^T$. From the exact sequence (3.4) we get that

$$Hom_A(A^s, A^t/Syz(N)) \cong Hom_A(A^s, A^t)/Im(d_N),$$

but

$$Im(d_N) \cong Im(d_N^*) = < I_s \otimes \Delta_N >,$$

where $< I_s \otimes \Delta_N >$ is the module generated by the columns of $I_s \otimes \Delta_N$.

Hence a presentation of $Hom_A(A^s, A^t/Syz(N))$ is

$$Hom_A(A^s, A^t/Syz(N)) \cong A^{ts}/ < I_s \otimes \Delta_N > . \tag{3.6}$$

This isomorphism is defined as follow. Let $f \in Hom_A(A^s, A^t/Syz(N))$ and $f(\boldsymbol{e}_i) = \overline{(f_{1i}, \ldots, f_{ti})}$, $1 \leq i \leq s$, then

$$Hom_A(A^s, A^t/Syz(N)) \xrightarrow{\theta_{s,t}} A^{ts}/ < I_s \otimes \Delta_N >$$

is defined by

$$\theta_{s,t}(f) = \overline{(f_{11}, \ldots, f_{t1}, \ldots, f_{1s}, \ldots, f_{ts})}.$$

We observe that $\theta_{s,t}$ is a surjective homomorphism. We have to prove that $\theta_{s,t}$ is injective. If $\theta_{s,t}(f) = \overline{\boldsymbol{0}}$, then $(f_{11}, \ldots, f_{t1}, \ldots, f_{1s}, \ldots, f_{ts}) \in$ $< I_s \otimes \Delta_N >$ and consequently

$$(f_{11}, \ldots, f_{t1}, \ldots, f_{1s}, \ldots, f_{ts}) = d_N^*(\boldsymbol{a})$$

for some $\boldsymbol{a} = (a_{11}, \dots, a_{t_1 1}, \dots, a_{1s}, \dots, a_{t_1 s})$. Thus,

$$\begin{aligned}(f_{11}, \dots, f_{t1}, \dots, f_{1s}, \dots, f_{ts}) &= \Phi_{s,t} \circ d_N \circ \Phi_{s,t_1}^{-1}(\boldsymbol{a}) \\ &= \Phi_{s,t} \circ d_N(a) \\ &= \Phi_{s,t}(\delta_N \circ a),\end{aligned}$$

where $a \in Hom_A(A^s, A^{t_1})$ is defined by $a(\boldsymbol{e}_i) = (a_{1i}, \dots, a_{t_1 i})$, $1 \leq i \leq s$. But for each $1 \leq i \leq s$, $(\delta_N \circ a)(\boldsymbol{e}_i) \in < \Delta_N >= Syz(N)$, hence $\overline{(f_{1i}, \dots, f_{ti})} = \overline{\boldsymbol{0}}$ for each $1 \leq i \leq s$. This means that $f = 0$.

In the same way, but using $Hom_A(A^{s_1}, \ )$, we obtain

$$Hom_A(A^{s_1}, A^t/Syz(N)) \cong A^{ts_1}/ < I_{s_1} \otimes \Delta_N >, \tag{3.7}$$

and the explicit isomorphism

$$Hom_A(A^{s_1}, A^t/Syz(N)) \xrightarrow{\theta_{s_1,t}} A^{ts_1}/ < I_{s_1} \otimes \Delta_N >$$

is defined by

$$\theta_{s_1,t}(h) = \overline{(h_{11}, \dots, h_{t1}, \dots, h_{1s_1}, \dots, h_{ts_1})},$$

where $h \in Hom_A(A^{s_1}, A^t/Syz(N))$.

In the example we have

$$< I_2 \otimes \Delta_N >= \begin{bmatrix} 5 & 2x+9y & 0 & 0 \\ 0 & 8x & 0 & 0 \\ 5 & y & 0 & 0 \\ 0 & 0 & 5 & 2x+9y \\ 0 & 0 & 0 & 8x \\ 0 & 0 & 5 & y \end{bmatrix}, < I_1 \otimes \Delta_N >= \begin{bmatrix} 5 & 2x+9y \\ 0 & 8x \\ 5 & y \end{bmatrix},$$

and hence

$$Hom_A(A^2, A^3/ < (5,0,5), (2x+9y, 8x, y) >) \cong A^6/ < I_2 \otimes \Delta_N >,$$
$$Hom_A(A, A^3/ < (5,0,5), (2x+9y, 8x, y) >) \cong A^3/ < I_1 \otimes \Delta_N >.$$

Step 4. Computing the matrix U. As in the previous step, the homomorphism δ_M in the sequence (3.1) induces the natural homomorphism d_M, and this one induces the homomorphism d_M^* defined by d_M and the natural vertical isomorphisms of the following commutative diagram

$$\begin{array}{ccc} Hom_A(A^s, A^t) & \xrightarrow{d_M} & Hom_A(A^{s_1}, A^t) \\ \Phi_{s,t} \big\downarrow & & \big\downarrow \Phi_{s_1,t} \\ A^{ts} & \xrightarrow{d_M^*} & A^{ts_1} \end{array}$$

Since $Hom_A(\ ,A^t)$ inverts the sense of arrows, in the canonical bases the matrix of d_M^* is

$$\Delta_M^T \otimes I_t = (\Delta_M \otimes I_t)^T,$$

where Δ_M^T is the transpose of the matrix Δ_M. In the example,

$$(\Delta_M \otimes I_3)^T = \begin{bmatrix} 5y & 0 & 0 & 5x & 0 & 0 \\ 0 & 5y & 0 & 0 & 5x & 0 \\ 0 & 0 & 5y & 0 & 0 & 5x \end{bmatrix}.$$

As in the equation (3.5), we can explicit the homomorphism d_M^*. Let $\boldsymbol{u} = (u_{11}, \dots, u_{t1}, \dots, u_{1s}, \dots, u_{ts})$, then $d_M^*(\boldsymbol{u}) = (\Delta_M \otimes I_t)^T \boldsymbol{u}^T$, and hence

$$d_M^*(\boldsymbol{u}) = (\sum_{k=1}^{s} u_{1k}\delta_{k1}, \dots, \sum_{k=1}^{s} u_{tk}\delta_{k1}, \dots, \sum_{k=1}^{s} u_{1k}\delta_{ks_1}, \dots, \sum_{k=1}^{s} u_{tk}\delta_{ks_1}) \tag{3.8}$$

where $\Delta_M = [\delta_{vz}]$.

We observe that $d_M^*(< I_s \otimes \Delta_N >) \subseteq < I_{s_1} \otimes \Delta_N >$. In fact, it is enough to prove that $d_M^*(\boldsymbol{u}) \in < I_{s_1} \otimes \Delta_N >$, where $\boldsymbol{u}$ is any column of $I_s \otimes \Delta_N$. But the form of $\boldsymbol{u}$ is

$$\boldsymbol{u} = (0, \dots, 0, \dots, \Delta_{1j}, \dots, \Delta_{tj}, \dots, 0, \dots, 0)$$

where $(\Delta_{1j}, \dots, \Delta_{tj})$ is the j-column of Δ_N, $1 \leq j \leq t_1$. We note that $(\Delta_{1j}, \dots, \Delta_{tj})$ can be located in s different places within $\boldsymbol{u}$, so we suppose that $(\Delta_{1j}, \dots, \Delta_{tj})$ is located in the l-position within $\boldsymbol{u}$, $1 \leq l \leq s$. Thus,

$$d_M^*(\boldsymbol{u}) = (\Delta_M \otimes I_t)^T \boldsymbol{u}^T = \begin{bmatrix} \delta_{l1}\Delta_{1j} \\ \vdots \\ \delta_{l1}\Delta_{tj} \\ \vdots \\ \delta_{ls_1}\Delta_{1j} \\ \vdots \\ \delta_{ls_1}\Delta_{tj} \end{bmatrix} = \delta_{l1} \begin{bmatrix} \Delta_{1j} \\ \vdots \\ \Delta_{tj} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \delta_{ls_1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Delta_{1j} \\ \vdots \\ \Delta_{tj} \end{bmatrix}$$

i.e., $d_M^*(\boldsymbol{u}) \in < I_{s_1} \otimes \Delta_N >$.

The homomorphism d_M^* induces the homomorphism $\overline{d_M^*}$ and we have the commutative diagram

$$\begin{array}{ccc} Hom_A(A^s, A^t) & \xrightarrow{d_M} & Hom_A(A^{s_1}, A^t) \\ \Phi_{s,t}\downarrow & & \downarrow\Phi_{s_1,t} \\ A^{ts} & \xrightarrow{d_M^*} & A^{ts_1} \\ j\downarrow & & \downarrow j_1 \\ A^{ts}/<I_s\otimes\Delta_N> & \xrightarrow{\overline{d_M^*}} & A^{ts_1}/<I_{s_1}\otimes\Delta_N> \end{array}$$

where j and j_1 are the canonical homomorphisms. We define the matrix U where its columns are the generators of $\ker(j_1\circ d_M^*)$, i.e., $<U>=\ker(j_1\circ d_M^*)$. We observe that the matrix of homomorphism $j_1\circ d_M^*$ coincides with the matrix of homomorphism d_M^*, then by the Theorem 8,

columns of U =first st coordinates of generators of

$$Syz\left([(\Delta_M\otimes I_t)^T|I_{s_1}\otimes\Delta_N)]\right)$$

(the enhanced matrix obtained from A adding the columns of B is denoted by $[A|B]$). For our illustrative example, we have computed $Syz([(\Delta_M\otimes I_3)^T|I_1\otimes\Delta_N)])$ using Theorem 4. Additionally, we computed a minimal Gröbner basis for $Syz([(\Delta_M\otimes I_3)^T|I_1\otimes\Delta_N)])$ and then we selected the first six coordinates of elements of this basis, the result of these computations was

$$U=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & x & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 1 & 0 & 0 \\ 0 & 9y & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}.$$

Step 5. Presentation of $Hom_A(A^s/Syz(M), A^t/Syz(N))$. From the above results we have the following commutative diagram

$$\begin{array}{ccc} Hom_A(A^s, A^t/Syz(N)) & \xrightarrow{d} & Hom_A(A^{s_1}, A^t/Syz(N)) \\ \theta_{s,t}\downarrow & & \downarrow\theta_{s_1,t} \\ A^{ts}/<I_s\otimes\Delta_N> & \xrightarrow{\overline{d_M^*}} & A^{ts_1}/<I_{s_1}\otimes\Delta_N>. \end{array}$$

In fact, let $f\in Hom_A(A^s, A^t/Syz(N))$ with matrix $F=[f_{ji}]$ given by $f(\boldsymbol{e}_i)=\overline{(f_{1i},\dots,f_{ti})}$, $1\le i\le s$. Then,

$$d(f) = f \circ \delta_M = h \in Hom_A(A^{s_1}, A^t/Syz(N)),$$

and hence,

$$\theta_{s_1,t}(h) = \overline{(\sum_{k=1}^s f_{1k}\delta_{k1}, \dots, \sum_{k=1}^s f_{tk}\delta_{k1}, \dots, \sum_{k=1}^s f_{1k}\delta_{ks_1}, \dots, \sum_{k=1}^s f_{tk}\delta_{ks_1})}.$$

On the other hand,

$$\begin{aligned}\overline{d_M^*}(\theta_{s,t}(f)) &= \overline{d_M^*}(\overline{(f_{11}, \dots, f_{t1}, \dots, f_{1s}, \dots, f_{ts})}) \\ &= \overline{d_M^*((f_{11}, \dots, f_{t1}, \dots, f_{1s}, \dots, f_{ts})} \\ &= \overline{(\Delta_M^T \otimes I_t)(f_{11}, \dots, f_{t1}, \dots, f_{1s}, \dots, f_{ts})} \\ &= \overline{(\sum_{k=1}^s f_{1k}\delta_{k1}, \dots, \sum_{k=1}^s f_{tk}\delta_{k1}, \dots, \sum_{k=1}^s f_{1k}\delta_{ks_1}, \dots, \sum_{k=1}^s f_{tk}\delta_{ks_1})}.\end{aligned}$$

Hence, $\ker(d) \cong \ker(\overline{d_M^*})$, and from (3.3), a presentation of $\ker(\overline{d_M^*})$ gives a presentation of $Hom_A(A^s/Syz(M), A^t/Syz(N))$. Let $\mathbf{u} \in A^{ts}$, then

$$\overline{\mathbf{u}} \in \ker(\overline{d_M^*}) \Longleftrightarrow \overline{d_M^*}(\overline{\mathbf{u}}) = \overline{\mathbf{0}} \Longleftrightarrow d_M^*(\mathbf{u}) \in < I_{s_1} \otimes \Delta_N > \Longleftrightarrow$$
$$(\Delta_M \otimes I_t)^T \boldsymbol{u}^T \in < I_{s_1} \otimes \Delta_N > \Longleftrightarrow$$
the first st coordiantes of $\mathbf{u}$ *belong to* $\mathrm{Syz}\left([(\Delta_M \otimes I_t)^T | I_{s_1} \otimes \Delta_N)]\right) \Longleftrightarrow$
$$\overline{\mathbf{u}} \in < U > / < I_s \otimes \Delta_N >,$$

where $< U >$ is the column module of matrix U. Thus, we have proved that $\ker(\overline{d_M^*}) = < U > / < I_s \otimes \Delta_N >$, and we get the following theorem.

Theorem 15. *With the notation above,*

$$Hom_A(M, N) \cong < U > / < I_s \otimes \Delta_N >, \tag{3.9}$$

and presentation of $< U > / < I_s \otimes \Delta_N >$ *is a presentation for* $Hom_A(M, N)$.

In the example we have $Hom_A(M, N) \cong A^8/K$, where K is the module generated by the first 8 entries of generators of $Syz([U|I_2 \otimes \Delta_N])$ (see the Theorem 7). We computed a minimal Gröbner basis for the syzygy of the matrix $[U|I_2 \otimes \Delta_N]$ and then we selected the first eight entries of the elements of this basis. The result was

$$Hom_A(M, N) \cong A^8/ < \boldsymbol{v}_1, \boldsymbol{v}_2, \mathbf{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6, \boldsymbol{v}_7, \boldsymbol{v}_8, \boldsymbol{v}_9 >$$

where

$$\begin{aligned}
\boldsymbol{v}_1 &=< 5,0,0,0,0,0,0,0 > \\
\boldsymbol{v}_2 &=< 8x+y,0,6x,0,x+4y,0,0,0 > \\
\boldsymbol{v}_3 &=< 0,0,5,0,0,0,0,0 > \\
\boldsymbol{v}_4 &=< 0,8,x,0,0,0,4y,0 > \\
\boldsymbol{v}_5 &=< 0,0,0,0,5,0,0,0 > \\
\boldsymbol{v}_6 &=< 0,0,0,8,x,2y,0,0 > \\
\boldsymbol{v}_7 &=< 0,0,0,0,0,0,5,0 > \\
\boldsymbol{v}_8 &=< 0,0,0,0,0,0,0,5 > \\
\boldsymbol{v}_9 &=< 0,0,0,0,0,4x+8y,8x,y > .
\end{aligned}$$

Step 6. We conclude this section computing an explicit set of generators for $Hom_A(M,N)$ in our illustrative example. We know that

$$Hom_A(M,N) \cong$$
$$Hom_A(A^2/ < (5y,5x) >, A^3/ < (5,0,5),(2x+9y,8x,y) >).$$

Each element $\phi \in Hom_A(M,N)$ can be represented by an unique element $\overline{\boldsymbol{a}} \in A^8/ < \boldsymbol{v}_1,\ldots,\boldsymbol{v}_9 >$ with $\boldsymbol{a} = a_1\boldsymbol{e}_1 + \cdots + a_8\boldsymbol{e}_8 \in A^8$. We consider the column i of U, $U\boldsymbol{e}_i$,$1 \le i \le 8$, the six entries of this column could be disposed into a 3×2 matrix, denoted by U_i, taking the first three entries as column one and the next three entries as column two. Thus, U_i represents an A-homomorphism defined by

$$A^2 \xrightarrow{U_i} A^3$$
$$\boldsymbol{z} \mapsto U_i\boldsymbol{z}$$

and it induces an A-homomorphism also denoted by U_i

$$A^2/K_M \xrightarrow{U_i} A^3/Syz(N)$$
$$\overline{\boldsymbol{z}} \mapsto \overline{U_i\boldsymbol{z}}.$$

It is easy to verify that each U_i is well defined, i.e. $U_i(K_M) \subseteq K_N$, $1 \le i \le 8$. The eight matrices are

$$U_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, U_2 = \begin{bmatrix} 0 & 0 \\ x & 9y \\ 0 & 0 \end{bmatrix} U_3 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}, U_4 = \begin{bmatrix} 0 & y \\ 0 & 0 \\ x & 0 \end{bmatrix}$$
$$U_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} U_6 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} U_7 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} U_8 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

The isomorphisms

$$A^2/K_M \cong M \qquad\qquad A^3/K_N \cong N$$
$$\overline{\boldsymbol{e}_i} \mapsto \boldsymbol{f}_i \qquad\qquad \overline{\boldsymbol{e}'_j} \mapsto \boldsymbol{g}_j$$

describe the eight homomorphisms of $Hom_A(M,N)$,

$$M \xrightarrow{\phi_1} N$$
$$\boldsymbol{f}_1 \mapsto \overline{\boldsymbol{e}_1} \mapsto \overline{U_1\boldsymbol{e}_1} = \overline{\boldsymbol{e}'_1} + \overline{\boldsymbol{e}'_3} = \boldsymbol{g}_1 + \boldsymbol{g}_3$$
$$\boldsymbol{f}_2 \mapsto \overline{\boldsymbol{e}_2} \mapsto \overline{U_1\boldsymbol{e}_2} = \boldsymbol{0}$$

$$M \xrightarrow{\phi_2} N \qquad M \xrightarrow{\phi_3} N \qquad M \xrightarrow{\phi_4} N$$
$$\boldsymbol{f}_1 \mapsto x\boldsymbol{g}_2 \qquad \boldsymbol{f}_1 \mapsto 2\boldsymbol{g}_2 \qquad \boldsymbol{f}_1 \mapsto x\boldsymbol{g}_3$$
$$\boldsymbol{f}_2 \mapsto 9y\boldsymbol{g}_2 \qquad \boldsymbol{f}_2 \mapsto \boldsymbol{0} \qquad \boldsymbol{f}_2 \mapsto y\boldsymbol{g}_1$$

$$M \xrightarrow{\phi_5} N \qquad M \xrightarrow{\phi_6} N \qquad M \xrightarrow{\phi_7} N$$
$$\boldsymbol{f}_1 \mapsto 2\boldsymbol{g}_3 \qquad \mathbf{f}_1 \mapsto \boldsymbol{0} \qquad \boldsymbol{f}_1 \mapsto \boldsymbol{0}$$
$$\boldsymbol{f}_2 \mapsto \boldsymbol{0} \qquad \boldsymbol{f}_2 \mapsto \boldsymbol{g}_1 + \boldsymbol{g}_3 \qquad \boldsymbol{f}_2 \mapsto 2\boldsymbol{g}_2$$

$$M \xrightarrow{\phi_8} N$$
$$\boldsymbol{f}_1 \mapsto \boldsymbol{0}$$
$$\boldsymbol{f}_2 \mapsto 2\boldsymbol{g}_3$$

3.2. **Computation of $M \otimes N$.** For $M = <\boldsymbol{f}_1, \ldots, \boldsymbol{f}_s> \subseteq A^m$ and $N = <\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t> \subseteq A^l$ we now compute a presentation of $M \otimes N$. This computation has been also considered in [8] using exact sequences (see [8], Corollary 2.7.8). Our presentation is given as a quotient module of $M \otimes N$ and showing an explicit set of generators for $Syz(M \otimes N)$. We start with a preliminary proposition (see [8], Proposition 2.7.10).

Proposition 16. *Let S be an arbitrary commutative ring and M, N modules over S. Let $\boldsymbol{m}_j \in M, \boldsymbol{g}_j \in N$, $1 \leq j \leq t$ such that $N = <\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t>$. Then, $\boldsymbol{m}_1 \otimes \boldsymbol{g}_1 + \cdots + \boldsymbol{m}_t \otimes \boldsymbol{g}_t = \boldsymbol{0}$ if and only if there exist elements $\boldsymbol{m}'_v \in M$ and $h_{jv} \in S$, such that $\boldsymbol{m}_j = \sum_{v=1}^{r} h_{jv}\boldsymbol{m}'_v$ and $\sum_{j=1}^{t} h_{jv}\boldsymbol{g}_j = \boldsymbol{0}$ for each $1 \leq v \leq r$, $1 \leq j \leq t$. In a matrix notation,*

$$[\boldsymbol{m}_1 \cdots \boldsymbol{m}_t] = [\boldsymbol{m}'_1 \cdots \boldsymbol{m}'_r]H^T, \; H^T[\boldsymbol{g}_1 \cdots \boldsymbol{g}_t] = 0,$$

where $H = [h_{jv}]$ *and the module expanded by the columns of* H *is contained in* $Syz(N)$.

Theorem 17. *Let* M *and* N *be submodules as above. Then,*

$$M \otimes N \cong A^{st}/Syz(M \otimes N)$$

where

$$Syz(M \otimes N) =< [Syz(M) \otimes I_t \mid I_s \otimes Syz(N)] > .$$

Proof. It is clear that

$$M \otimes N =< \boldsymbol{f}_1 \otimes \boldsymbol{g}_1, \ldots, \boldsymbol{f}_1 \otimes \boldsymbol{g}_t, \ldots, \boldsymbol{f}_s \otimes \boldsymbol{g}_1, \ldots, \boldsymbol{f}_s \otimes \boldsymbol{g}_t > . \quad (3.10)$$

Let $Syz(M) =< \boldsymbol{f}'_1, \ldots, \boldsymbol{f}'_r >$ and $Syz(N) =< \boldsymbol{g}'_1, \ldots, \boldsymbol{g}'_p >$, with

$$\boldsymbol{f}'_1 = (f_{11}, \ldots, f_{s1}), \ldots, \boldsymbol{f}'_r = (f_{1r}, \ldots, f_{sr})$$

and

$$\boldsymbol{g}'_1 = (g_{11}, \ldots, g_{t1}), \ldots, \boldsymbol{g}'_p = (g_{1p}, \ldots, g_{tp}).$$

In a matrix notation,

$$Syz(M) = \begin{bmatrix} f_{11} & \cdots & f_{1r} \\ \vdots & \cdots & \vdots \\ f_{s1} & \cdots & f_{sr} \end{bmatrix}, \; Syz(N) = \begin{bmatrix} g_{11} & \cdots & g_{1p} \\ \vdots & \cdots & \vdots \\ g_{t1} & \cdots & g_{tp} \end{bmatrix}.$$

Then,

$$\begin{aligned} f_{11}\boldsymbol{f}_1 + \cdots + f_{s1}\boldsymbol{f}_s &= \boldsymbol{0} \\ &\vdots \\ f_{1r}\boldsymbol{f}_1 + \cdots + f_{sr}\boldsymbol{f}_s &= \boldsymbol{0} \end{aligned}$$

and

$$\begin{aligned} g_{11}\boldsymbol{g}_1 + \cdots + g_{t1}\boldsymbol{g}_t &= \boldsymbol{0} \\ &\vdots \\ g_{1p}\boldsymbol{g}_1 + \cdots + g_{tp}\boldsymbol{g}_t &= \boldsymbol{0}. \end{aligned}$$

We note that any of the following tr vectors has st entries and belongs to $Syz(M \otimes N)$

$$\begin{gathered}
(f_{11},0,\dots,0,\dots,f_{s1},0,\dots,0)\\
\vdots\\
(0,0,\dots,f_{11},\dots,0,0,\dots,f_{s1})\\
\vdots\\
(f_{1r},0,\dots,0,\dots,f_{sr},0,\dots,0)\\
\vdots\\
(0,0,\dots,f_{1r},\dots,0,0,\dots,f_{sr}).
\end{gathered}$$

In the same way, any of the following ps vectors has st entries and belongs to $Syz(M\otimes N)$

$$\begin{gathered}
(g_{11},\dots,g_{t1},\dots,0,\dots,0)\\
\ddots\\
(0,\dots,0,\dots,g_{11},\dots,g_{t1})\\
\vdots\\
(g_{1p},\dots,g_{tp},\dots,0,\dots,0)\\
\ddots\\
(0,\dots,0,\dots,g_{1p},\dots,g_{tp}).
\end{gathered}$$

We can dispose these $tr+ps$ vectors by columns in a matrix $[C \mid B]$ of size $st\times(tr+ps)$, where

$$C=\begin{bmatrix}
f_{11} & \dots & 0 & \dots & f_{1r} & \dots & 0\\
 & \ddots & & \dots & & \ddots & \\
0 & \dots & f_{11} & \dots & 0 & \dots & f_{1r}\\
 & \vdots & & \dots & & \vdots & \\
f_{s1} & \dots & 0 & \dots & f_{sr} & \dots & 0\\
 & \ddots & & \dots & & \ddots & \\
0 & \dots & f_{s1} & \dots & 0 & \dots & f_{sr}
\end{bmatrix}=Syz(M)\otimes I_t$$

$$B=\begin{bmatrix}
g_{11} & \dots & 0 & \dots & g_{1p} & \dots & 0\\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\
g_{t1} & \dots & 0 & \dots & g_{tp} & \dots & 0\\
\vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots\\
0 & \dots & g_{11} & \dots & 0 & \dots & g_{1p}\\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\
0 & \dots & g_{t1} & \dots & 0 & \dots & g_{tp}
\end{bmatrix}.$$

But B can be changed by

$$\begin{bmatrix} g_{11} & \dots & g_{1p} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ g_{t1} & \dots & g_{tp} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & g_{11} & \dots & g_{1p} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & g_{t1} & \dots & g_{tp} \end{bmatrix} = I_s \otimes Syz(N).$$

Thus, we have proved that $< [Syz(M) \otimes I_t \mid I_s \otimes Syz(N)] > \subseteq Syz(M \otimes N)$.

Now we assume that $\boldsymbol{h} = (h_{11}, \dots, h_{1t}, \dots, h_{s1}, \dots, h_{st}) \in Syz(M \otimes N)$, then

$$h_{11}(\boldsymbol{f}_1 \otimes \boldsymbol{g}_1) + \dots + h_{1t}(\boldsymbol{f}_1 \otimes \boldsymbol{g}_t) + \dots + h_{s1}(\boldsymbol{f}_s \otimes \boldsymbol{g}_1) + \dots + h_{st}(\boldsymbol{f}_s \otimes \boldsymbol{g}_t) = \mathbf{0}.$$

From this we get that

$$\boldsymbol{m}_1 \otimes \boldsymbol{g}_1 + \dots + \boldsymbol{m}_t \otimes \boldsymbol{g}_t = \mathbf{0},$$

where

$$\boldsymbol{m}_j = h_{1j}\boldsymbol{f}_1 + \dots + h_{sj}\boldsymbol{f}_s \in M,$$

with $1 \leq j \leq t$. From the Proposition 16, there exist polynomials $a_{jv} \in A$ and vectors $\boldsymbol{m}'_v \in M$ such that $\boldsymbol{m}_j = \sum_{v=1}^{r} a_{jv}\boldsymbol{m}'_v$ and $\sum_{j=1}^{t} a_{jv}\boldsymbol{g}_j = \mathbf{0}$ for each $1 \leq v \leq r$. This means that $(a_{1v}, \dots, a_{tv}) \in Syz(N)$ for each $1 \leq v \leq r$. Since $\boldsymbol{m}'_v \in M$ there exist $q_{uv} \in A$ such that $\boldsymbol{m}'_v = q_{1v}\boldsymbol{f}_1 + \dots + q_{sv}\boldsymbol{f}_s$, and then

$$\begin{aligned} \textstyle\sum_{i=1}^{s} h_{i1}\boldsymbol{f}_i &= a_{11}(q_{11}\boldsymbol{f}_1 + \dots + q_{s1}\boldsymbol{f}_s) + \dots + a_{1r}(q_{1r}\boldsymbol{f}_1 + \dots + q_{sr}\boldsymbol{f}_s) \\ &\vdots \\ \textstyle\sum_{i=1}^{s} h_{it}\boldsymbol{f}_i &= a_{t1}(q_{11}\boldsymbol{f}_1 + \dots + q_{s1}\boldsymbol{f}_s) + \dots + a_{tr}(q_{1r}\boldsymbol{f}_1 + \dots + q_{sr}\boldsymbol{f}_s). \end{aligned}$$

From this we get that

$$\begin{aligned} \textstyle\sum_{i=1}^{s}(h_{i1} - (a_{11}q_{i1} + \dots + a_{1r}q_{ir}))\boldsymbol{f}_i &= \mathbf{0} \\ &\vdots \\ \textstyle\sum_{i=1}^{s}(h_{it} - (a_{t1}q_{i1} + \dots + a_{tr}q_{ir}))\boldsymbol{f}_i &= \mathbf{0} \end{aligned}$$

i.e.,

$$(h_{11}-(a_{11}q_{11}+\cdots+a_{1r}q_{1r}),\ldots,h_{s1}-(a_{11}q_{s1}+\cdots+a_{1r}q_{sr}))\in Syz(M)$$
$$\vdots$$
$$(h_{1t}-(a_{t1}q_{11}+\cdots+a_{tr}q_{1r}),\ldots,h_{st}-(a_{t1}q_{s1}+\cdots+a_{tr}q_{sr}))\in Syz(M).$$

This implies that

$$(h_{11},\ldots,h_{s1})-(a_{11}q_{11}+\cdots+a_{1r}q_{1r},\ldots,a_{11}q_{s1}+\cdots+a_{1r}q_{sr})\in Syz(M)$$
$$\vdots$$
$$(h_{1t},\ldots,h_{st})-(a_{t1}q_{11}+\cdots+a_{tr}q_{1r},\ldots,a_{t1}q_{s1}+\cdots+a_{tr}q_{sr})\in Syz(M).$$

Then,

$$(h_{i1})_{i=1}^{s}=(a_{11}q_{11}+\cdots+a_{1r}q_{1r},\ldots,a_{11}q_{s1}+\cdots+a_{1r}q_{sr})+(f_{11},\ldots,f_{s1})$$
$$\vdots$$
$$(h_{it})_{i=1}^{s}=(a_{t1}q_{11}+\cdots+a_{tr}q_{1r},\ldots,a_{t1}q_{s1}+\cdots+a_{tr}q_{sr})+(f_{1t},\ldots,f_{st}),$$

with $(f_{11},\ldots,f_{s1}),\ldots,(f_{1t},\ldots,f_{st})\in Syz(M)$. From this we get

$$\begin{gathered}
h_{11}=f_{11}+a_{11}q_{11}+\cdots+a_{1r}q_{1r}\\
\vdots\\
h_{1t}=f_{1t}+a_{t1}q_{11}+\cdots+a_{tr}q_{1r}\\
\vdots\\
h_{s1}=f_{s1}+a_{11}q_{s1}+\cdots+a_{1r}q_{sr}\\
\vdots\\
h_{st}=f_{st}+a_{t1}q_{s1}+\cdots+a_{tr}q_{sr},
\end{gathered}$$

and hence $\boldsymbol{h}$ is a linear combination of columns of the following matrix

$$[Syz(M)\otimes I_t \mid D]$$

where

$$D=\begin{bmatrix}
a_{11} & \ldots & a_{1r} & \ldots & 0 & \ldots & 0\\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots\\
a_{t1} & \ldots & a_{tr} & \ldots & 0 & \ldots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & \ldots & 0 & \ldots & a_{11} & \ldots & a_{1r}\\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots\\
0 & \ldots & 0 & \ldots & a_{t1} & \ldots & a_{tr}
\end{bmatrix}.$$

But $< D > \subseteq < I_s \otimes Syz(N) >$, and hence, $h \in < [Syz(M) \otimes I_t \mid I_s \otimes Syz(N)] >$. This completes the proof of the proposition. □

Example 18. Let M and N be submodules as in the Example 6, then

$$Syz(M \otimes N) = [Syz(M) \otimes I_3 \mid I_2 \otimes Syz(N)],$$

i.e.,

$$Syz(M \otimes N) = \begin{bmatrix} 5y & 0 & 0 & 5 & 2x+9y & 0 & 0 \\ 0 & 5y & 0 & 0 & 8x & 0 & 0 \\ 0 & 0 & 5y & 5 & y & 0 & 0 \\ 5x & 0 & 0 & 0 & 0 & 5 & 2x+9y \\ 0 & 5x & 0 & 0 & 0 & 0 & 8x \\ 0 & 0 & 5x & 0 & 0 & 5 & y \end{bmatrix}.$$

3.3. **Computing free resolutions.** In this subsection we will compute free resolutions for submodules of A^m. Let M be a submodule of A^m, we recall that a free resolution of M is an exact sequence of free modules

$$\cdots \xrightarrow{F_{r+2}} A^{s_r} \xrightarrow{F_r} A^{s_{r-1}} \xrightarrow{F_{r-1}} \cdots \xrightarrow{F_2} A^{s_1} \xrightarrow{F_1} A^{s_0} \xrightarrow{F_0} M \longrightarrow 0,$$

with $s_i \geq 0$ for each $i \geq 0$. We assume that $A^0 = 0$. The following proposition describes a simple procedure for constructing a free resolution of M.

Theorem 19. *Let $M =< \boldsymbol{f}_1^{(0)}, \ldots, \boldsymbol{f}_{s_0}^{(0)} >$ be a submodule of the free module A^m. Let F_0 be the matrix whose columns are $\boldsymbol{f}_1^{(0)}, \ldots, \boldsymbol{f}_{s_0}^{(0)}$, and for $i \geq 1$ let*

$$F_i = Syz(F_{i-1}) = [\boldsymbol{f}_1^{(i)} \cdots \boldsymbol{f}_{s_i}^{(i)}].$$

Then,

$$\cdots \xrightarrow{F_r+2} A^{s_r} \xrightarrow{F_r} A^{s_{r-1}} \xrightarrow{F_{r-1}} \cdots \xrightarrow{F_2} A^{s_1} \xrightarrow{F_1} A^{s_0} \xrightarrow{F_0} M \longrightarrow 0,$$

is a free resolution of M, where

$$F_i \boldsymbol{e}_{j_i}^{(i)} = \boldsymbol{f}_{j_i}^{(i)}$$

and $\{\boldsymbol{e}_{j_i}^{(i)}\}_{1 \leq j_i \leq s_i}$ is the canonical basis of A^{s_i}.

Proof. Each homomorphism F_i is represented by a matrix, and hence, a resolution of M is described as a sequence of matrices $\{F_i\}_{i \geq 0}$, where the columns of F_i are the generators of $Syz(F_{i-1})$, $i \geq 1$. The columns of F_0 are the generators of M. We note that $Im(F_i) = Syz(F_{i-1}) = \ker(F_{i-1})$ for each $i \geq 1$, and moreover F_0 is a surjective homomorphism. □

We can illustrate this procedure in the following example.

Example 20. Let M and N be as in the Example 6, we will compute free resolutions for M and N. According to the Proposition 19, we must compute the matrices F_i, $i \geq 0$, and for this, we will use the Theorem 4.

We start with M, in this case we have

$$F_0 = [\boldsymbol{f}_1 \quad \boldsymbol{f}_2] = \begin{bmatrix} 3x^2y + 3x & 7xy^2 + y \\ xy - 2y & y^2 - 4x \end{bmatrix},$$

$$F_1 = Syz(F_0) = \begin{bmatrix} 5y \\ 5x \end{bmatrix}, F_2 = Syz(F_1) = [2], F_3 = Syz(F_2) = [5], \ldots$$

hence, for $r \geq 1$

$$F_{2r} = [2], F_{2r+1} = [5].$$

Thus, a free resolution for M is

$$\cdots \to A \xrightarrow{[5]} A \xrightarrow{[2]} A \xrightarrow{\begin{bmatrix} 5y \\ 5x \end{bmatrix}} A^2 \xrightarrow{\begin{bmatrix} 3x^2y + 3x & 7xy^2 + y \\ xy - 2y & y^2 - 4x \end{bmatrix}} M \to 0.$$

For N we have

$$G_0 = [\boldsymbol{g}_1 \quad \boldsymbol{g}_2 \quad \boldsymbol{g}_3] = \begin{bmatrix} 0 & y & 2x \\ x & x & x \end{bmatrix}, G_1 = Syz(G_0) = \begin{bmatrix} 5 & 2x + 9y \\ 0 & 8x \\ 5 & y \end{bmatrix},$$

$$G_2 = Syz(G_1) = \begin{bmatrix} 2 & y \\ 0 & 5 \end{bmatrix}, G_3 = Syz(G_2) = \begin{bmatrix} 5 & y \\ 0 & 8 \end{bmatrix},$$

$$G_4 = Syz(G_3) = \begin{bmatrix} 2 & y \\ 0 & 5 \end{bmatrix}, \ldots$$

hence, for $r \geq 1$

$$G_{2r} = \begin{bmatrix} 2 & y \\ 0 & 5 \end{bmatrix}, G_{2r+1} = \begin{bmatrix} 5 & y \\ 0 & 8 \end{bmatrix}.$$

Thus, a free resolution for N is

$$\cdots \to A^2 \xrightarrow{\begin{bmatrix} 5 & y \\ 0 & 8 \end{bmatrix}} A^2 \xrightarrow{\begin{bmatrix} 2 & y \\ 0 & 5 \end{bmatrix}} A^2$$

$$A^2 \xrightarrow{\begin{bmatrix} 5 & 2x + 9y \\ 0 & 8x \\ 5 & y \end{bmatrix}} A^3 \xrightarrow{\begin{bmatrix} 0 & y & 2x \\ x & x & x \end{bmatrix}} N \to 0$$

4. **Computation of** Ext **and** Tor

Now we will present the main computations of this paper, the modules $Ext^r_A(M)$ and $Tor^A_r(M,N)$.

4.1. **Computation of** Ext**.** Using syzygies and the results of previous sections we now describe an easy procedure for computing the A-modules $Ext^r_A(M,N)$ for $r \geq 0$, where $M = < \boldsymbol{f}_1, \dots, \boldsymbol{f}_s > \subseteq A^m$ and $N =< \boldsymbol{g}_1, \dots, \boldsymbol{g}_t > \subseteq A^l$. By computing we mean to find a presentation and a system of generators of $Ext^r_A(M,N)$ (compare with [1], Section 3.10). For $r = 0$, $Ext^0_A(M,N) = Hom_A(M,N)$ and the computation is given by the results of Section 3. So we assume that $r \geq 1$.

Presentation of $Ext^r_A(M,N)$, $r \geq 1$:

Step 1. We compute presentations of M and N,

$$M \cong A^s/Syz(M), N \cong A^t/Syz(N),$$

and we know how to compute $Syz(M)$ and $Syz(N)$.

Step 2. Using Theorem 19 we compute a free resolution of $A^s/Syz(M)$,

$$\cdots \xrightarrow{F_{r+2}} A^{s_{r+1}} \xrightarrow{F_{r+1}} A^{s_r} \xrightarrow{F_r} A^{s_{r-1}} \xrightarrow{F_{r-1}} \cdots \xrightarrow{F_2} A^{s_1} \xrightarrow{F_1} A^{s_0} \xrightarrow{F_0} A^s/Syz(M) \longrightarrow 0.$$

Thus, using syzygies we can compute the matrices F_r, for $r \geq 1$.

Step 3. We consider the complex

$$0 \longrightarrow Hom_A(A^{s_0}, A^t/Syz(N)) \xrightarrow{F_1^*} \cdots \xrightarrow{F_r^*} Hom_A(A^{s_r}, A^t/Syz(N)) \xrightarrow{F_{r+1}^*} Hom_A(A^{s_{r+1}}, A^t/Syz(N)) \xrightarrow{F_{r+2}^*} \cdots$$

and we recall that

$$Ext^r_A(M,N) \cong Ext^r_A(A^s/Syz(M), A^t/Syz(N)) = \ker(F_{r+1}^*)/Im(F_r^*). \tag{4.1}$$

However, by (3.6), for each $r \geq 1$ a presentation of $Hom_A(A^{s_r}, A^t/Syz(N))$ is

$$Hom_A(A^{s_r}, A^t/Syz(N)) \cong A^{ts_r}/ < I_{s_r} \otimes Syz(N) >.$$

Step 4. We compute the matrices F^* in the following way: as we saw in (2.2), we have the following commutative diagram

$$\begin{array}{ccc} Hom_A(A^{s_r}, A^t/Syz(N)) & \xrightarrow{F^*_{r+1}} & Hom_A(A^{s_{r+1}}, A^t/Syz(N)) \\ \downarrow & & \downarrow \\ A^{ts_r}/K_r & \xrightarrow{\overline{F^*_{r+1}}} & A^{ts_{r+1}}/K_{r+1} \end{array}$$

where $K_r =< I_{s_r} \otimes Syz(N) >$ and vertical arrows are isomorphisms obtained concatenating the columns of matrices of $Hom_A(A^{s_r}, A^t/Syz(N))$, moreover, the matrices of homomorphisms F^*_{r+1} and $\overline{F^*_{r+1}}$ coincides. So, we can replace the above complex for the following equivalent complex

$$0 \longrightarrow A^{ts_0}/K_0 \xrightarrow{F_1^*} \cdots \xrightarrow{F_r^*} A^{ts_r}/K_r \xrightarrow{F^*_{r+1}} A^{ts_{r+1}}/K_{r+1} \xrightarrow{F^*_{r+2}} \cdots.$$

We will compute the matrix F^*_{r+1}: let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{ts_r}\}$ be the canonical basis of A^{ts_r}, then for each $1 \leq i \leq ts_r$, the element $\overline{\boldsymbol{e}_i} = \boldsymbol{e}_i + K_r$ can be replaced by its corresponding canonical matrix G_i, and since $F^*_{r+1}(G) = GF_{r+1}$ for each $G \in Hom_A(A^{s_r}, A^t/Syz(N))$, then we conclude that

$$F^*_{r+1} = I_t \otimes F^T_{r+1}.$$

Step 5. By (4.1), a presentation of $Ext^r_A(M, N)$ is given by a presentation of $\ker(F^*_{r+1})/Im(F^*_r)$. Hence, we can apply the Theorem 7, let p_r be the number of generators of $\ker(F^*_{r+1}) = Syz(F^*_{r+1}) = Syz(I_t \otimes F^T_{r+1})$, we know how to compute this syzygy, and hence, we know how to compute p_r. We also know how to compute the matrix $F^*_r = I_t \otimes F^T_r$. Then, a presentation of $Ext^r_A(M, N)$ is given by

$$Ext^r_A(M, N) \cong A^{p_r}/Syz(\ker(F^*_{r+1})/Im(F^*_r)), \tag{4.2}$$

where a set of generators of $Syz(\ker(F^*_{r+1})/Im(F^*_r))$ are the first p_r coordinates of the generators of

$$Syz[\ker(F^*_{r+1})|Im(F^*_r))] = Syz[Syz[I_t \otimes F^T_{r+1}]|I_t \otimes F^T_r]. \tag{4.3}$$

System of generators of $Ext^r_A(M, N)$, $r \geq 1$: By (4.1), a system of generators for $Ext^r_A(M, N)$ is defined by a system of generators of $\ker(F^*_{r+1}) = Syz(F^*_{r+1}) = Syz(I_t \otimes F^T_{r+1})$, hence if

$$Syz[I_t \otimes F^T_{r+1}] = [\boldsymbol{h}_1 \cdots \boldsymbol{h}_{p_r}],$$

then

$$Ext^r_A(M, N) =< \widetilde{\boldsymbol{h}_1}, \ldots, \widetilde{\boldsymbol{h}_{p_r}} >,$$

where $\widetilde{\boldsymbol{h}_u} = \boldsymbol{h}_u + Im(F_r^*)$, $1 \leq u \leq p_r$.

In the next example we will illustrate these procedures.

Example 21. Let M and N be submodules as in the Example 6, we will compute $Ext_M^r(M,N)$, for $r \geq 1$. We will use the free resolution of M that we computed in the Example 20. For $r = 1$, we compute

$$Syz[I_3 \otimes F_2^T] = Syz \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

and hence, the value of p_1 in (4.2) is 3. Moreover,

$$I_3 \otimes F_1^T = \begin{bmatrix} 5y & 5x & 0 & 0 & 0 & 0 \\ 0 & 0 & 5y & 5x & 0 & 0 \\ 0 & 0 & 0 & 0 & 5y & 5x \end{bmatrix}.$$

Next we must compute

$$\begin{aligned} S_1 &= Syz[Syz[I_3 \otimes F_2^T] | I_3 \otimes F_1^T] = \\ &= Syz \begin{bmatrix} 5 & 0 & 0 & 5y & 5x & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 5y & 5x & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 5y & 5x \end{bmatrix}, \end{aligned}$$

and we get that

$$S_1 = \begin{bmatrix} 2 & 9y & 9x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 9y & 9x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 9y & 9x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 9y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 9y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 9y \end{bmatrix}.$$

In this matrix we choose the first three entries of each column and we get the generators of S_1, moreover, since 9 is invertible in $\mathbb{Z}_{10}$, then

$$\begin{aligned} \boldsymbol{u}_1 &= (2,0,0), \boldsymbol{u}_2 = (y,0,0), \boldsymbol{u}_3 = (x,0,0) \\ \boldsymbol{u}_4 &= (0,2,0), \boldsymbol{u}_5 = (0,y,0), \boldsymbol{u}_6 = (0,x,0) \\ \boldsymbol{u}_7 &= (0,0,2), \boldsymbol{u}_8 = (0,0,y), \boldsymbol{u}_9 = (0,0,x). \end{aligned}$$

Thus,

$$Ext_A^1(M,N) \cong A^3 / < \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5, \boldsymbol{u}_6, \boldsymbol{u}_7, \boldsymbol{u}_8, \boldsymbol{u}_9 >.$$

Moreover,

$$Ext^1_A(M,N) =< \widetilde{\boldsymbol{h}_1}, \widetilde{\boldsymbol{h}_2}, \widetilde{\boldsymbol{h}_3} >,$$

where

$$\widetilde{\boldsymbol{h}_1} = 5\boldsymbol{e}_1 + < [I_3 \otimes F_1^T] >, \widetilde{\boldsymbol{h}_2} = 5\boldsymbol{e}_2 + < [I_3 \otimes F_1^T] >,$$
$$\widetilde{\boldsymbol{h}_3} = 5\boldsymbol{e}_3 + < [I_3 \otimes F_1^T] >.$$

For $r = 2$ we have

$$Syz[I_3 \otimes F_3^T] = Syz \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

so $p_2 = 3$, moreover,

$$I_3 \otimes F_2^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

and hence $\ker(F_3^*) = Im(F_2^*)$, i.e.,

$$Ext^2_A(M,N) = 0.$$

We can check this result using (4.3), thus we compute

$$S_2 = Syz[Syz[I_3 \otimes F_3^T] | I_3 \otimes F_2^T] = Syz \begin{bmatrix} 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{bmatrix},$$

and we get that

$$S_2 = \begin{bmatrix} 5 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 5 \end{bmatrix},$$

thus the generators of S_2 are

$$\boldsymbol{v}_1 = (5,0,0), \boldsymbol{v}_2 = (9,0,0),$$
$$\boldsymbol{v}_3 = (0,5,0), \boldsymbol{v}_4 = (0,9,0),$$
$$\boldsymbol{v}_5 = (0,0,5), \boldsymbol{v}_6 = (0,0,9).$$

But, $< \boldsymbol{v}_2, \boldsymbol{v}_4, \boldsymbol{v}_6 >= A^3$, and hence, $Ext^2_A(M,N) \cong A^3/ < \boldsymbol{v}_1, \dots, \boldsymbol{v}_6 >= 0$. Moreover, $Ext^2_A(M,N) =< \widetilde{\boldsymbol{l}_1}, \widetilde{\boldsymbol{l}_2}, \widetilde{\boldsymbol{l}_3} >$, where $\widetilde{\boldsymbol{l}_1} = 2\boldsymbol{e}_1 + < [I_3 \otimes F_2^T] >= \widetilde{0}$, $\widetilde{\boldsymbol{l}_2} = 2\boldsymbol{e}_2 + < [I_3 \otimes F_2^T] >= \widetilde{0}$ and $\widetilde{\boldsymbol{l}_3} = 2\boldsymbol{e}_3 + < [I_3 \otimes F_2^T] >= \widetilde{0}$.

For $r = 3$ we have

$$Syz[I_3 \otimes F_4^T] = Syz \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

so $p_2 = 3$, moreover,

$$I_3 \otimes F_3^T = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

and hence $\ker(F_4^*) = Im(F_3^*)$, i.e.,

$$Ext_A^3(M, N) = 0.$$

We can check this result using (4.3), thus we compute

$$S_3 = Syz[Syz[I_3 \otimes F_4^T] | I_3 \otimes F_3^T] = Syz \begin{bmatrix} 5 & 0 & 0 & 5 & 0 & 0 \\ 0 & 5 & 0 & 0 & 5 & 0 \\ 0 & 0 & 5 & 0 & 0 & 5 \end{bmatrix},$$

and we get that

$$S_3 = \begin{bmatrix} 2 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix},$$

thus the generators of S_3 are

$$\boldsymbol{w}_1 = (2, 0, 0), \boldsymbol{w}_2 = (9, 0, 0),$$
$$\boldsymbol{w}_3 = (0, 2, 0), \boldsymbol{w}_4 = (0, 9, 0),$$
$$\boldsymbol{w}_5 = (0, 0, 2), \boldsymbol{w}_6 = (0, 0, 9).$$

But, $< \boldsymbol{w}_2, \boldsymbol{w}_4, \boldsymbol{w}_6 >= A^3$, and hence, $Ext_A^3(M, N) \cong A^3 / < \boldsymbol{w}_1, \ldots, \boldsymbol{w}_6 > = 0$. Moreover, $Ext_A^3(M, N) =< \widetilde{\boldsymbol{m}_1}, \widetilde{\boldsymbol{m}_2}, \widetilde{\boldsymbol{m}_3} >$, where $\widetilde{\boldsymbol{m}_1} = 5\boldsymbol{e}_1 + < [I_3 \otimes F_3^T] >= \widetilde{0}$, $\widetilde{\boldsymbol{m}_2} = 5\boldsymbol{e}_2 + < [I_3 \otimes F_3^T] >= \widetilde{0}$ and $\widetilde{\boldsymbol{m}_3} = 5\boldsymbol{e}_3 + < [I_3 \otimes F_3^T] >= \widetilde{0}$.

We conclude that

$$Ext_A^r(M, N) = 0 \text{ , for } r \geq 2.$$

4.2. **Computation of Tor.** Using syzygies and some previous results we now describe an easy procedure for computing the A-modules $Tor_r^A(M, N)$ for $r \geq 0$, where $M =< \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s > \subseteq A^m$ and $N =< \boldsymbol{g}_1, \ldots, \boldsymbol{g}_t > \subseteq A^l$.

By computing we mean to find a presentation and a system of generators of $Tor_r^A(M,N)$, $r \geq 0$ (compare with [8], Proposition 7.1.3). For $r = 0$, the computation is given by the Theorem 17. So we assume that $r \geq 1$.

Presentation of $Tor_r^A(M,N)$, $r \geq 1$:

Step 1. We compute presentations of M and N,

$$M \cong A^s/Syz(M), N \cong A^t/Syz(N),$$

Step 2. We compute a free resolution of $A^t/Syz(N)$ using theorem 19,

$$\cdots \xrightarrow{G_{r+2}} A^{t_{r+1}} \xrightarrow{G_{r+1}} A^{t_r} \xrightarrow{G_r} A^{t_{r-1}} \xrightarrow{G_{r-1}} \cdots \xrightarrow{G_2} A^{t_1} \xrightarrow{G_1} A^{t_0} \xrightarrow{G_0} A^t/Syz(N) \longrightarrow 0.$$

Step 3. We consider the complex

$$\cdots \xrightarrow{i\otimes G_{r+2}} A^s/Syz(M) \otimes A^{t_{r+1}} \xrightarrow{i\otimes G_{r+1}} A^s/Syz(M) \otimes A^{t_r} \xrightarrow{i\otimes G_r} \cdots \xrightarrow{i\otimes G_2} A^s/Syz(M) \otimes A^{t_1} \xrightarrow{i\otimes G_1} A^s/Syz(M) \otimes A^{t_0} \longrightarrow 0,$$

where i is the identical of $A^s/Syz(M)$ and then

$$Tor_r^A(M,N) \cong Tor_r^A(A^s/Syz(M), A^t/Syz(N)) = \ker(1 \otimes G_r)/Im(1 \otimes G_{r+1}),$$

but the matrix of $i \otimes G_r$ is $I_s \otimes G_r$, so $\ker(i \otimes G_r) = \ker(I_s \otimes G_r) = Syz(I_s \otimes G_r)$ and we get

$$Tor_r^A(M,N) \cong \ker(I_s \otimes G_r)/Im(I_s \otimes G_{r+1}). \tag{4.4}$$

Step 3. Let q_r be the number of generators of $Syz(I_s \otimes G_r)$, then by the Theorem 7, a presentation of $Tor_r^A(M,N)$ is given by

$$Tor_r^A(M,N) \cong A^{q_r}/Syz(\ker(I_s \otimes G_r)/Im(I_s \otimes G_{r+1})), \tag{4.5}$$

where a set of generators of $Syz(\ker(I_s \otimes G_r)/Im(I_s \otimes G_{r+1}))$ are the first q_r coordinates of generators of

$$Syz[Syz[I_s \otimes G_r]|I_s \otimes G_{r+1}]. \tag{4.6}$$

System of generators of $Tor_r^A(M,N)$, $r \geq 1$: By (4.4), a system of generators of $Tor_r^A(M,N)$ is given by a system of generators of $\ker(i \otimes G_r) = Syz(I_s \otimes G_r)$. Thus, if

$$Syz[I_s \otimes G_r] = [\boldsymbol{h}_1 \cdots \boldsymbol{h}_{q_r}],$$

then

$$Tor_r^A(M,N) = < \widetilde{\boldsymbol{h}_1}, \ldots, \widetilde{\boldsymbol{h}_{q_r}} >,$$

where $\widetilde{\boldsymbol{h}_v} = \boldsymbol{h}_v + Im(I_s \otimes FG_{r+1})$, $1 \le v \le q_r$.

In the next example we will illustrate these procedures.

Example 22. Let M and N be submodules as in the Example 6, we will compute $Tor_r^A(M, N)$, for $r \ge 1$. We will use the free resolution of N that we computed in the Example 20. For $r = 1$, we compute

$$\ker(I_2 \otimes G_1) = Syz[I_2 \otimes G_1] = Syz \begin{bmatrix} 5 & 2x+9y & 0 & 0 \\ 0 & 8x & 0 & 0 \\ 5 & y & 0 & 0 \\ 0 & 0 & 5 & 2x+9y \\ 0 & 0 & 0 & 8x \\ 0 & 0 & 5 & y \end{bmatrix} =$$

$$\begin{bmatrix} 2 & y & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & y \\ 0 & 0 & 0 & 5 \end{bmatrix},$$

but

$$I_2 \otimes G_2 = \begin{bmatrix} 2 & y & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & y \\ 0 & 0 & 0 & 5 \end{bmatrix},$$

hence $\ker(I_2 \otimes G_1) = Im(I_2 \otimes G_2)$, and by (4.4),

$$Tor_1^A(M, N) = 0.$$

For $r = 2$ we have

$$\ker(I_2 \otimes G_2) = Syz[I_2 \otimes G_2] = Syz \begin{bmatrix} 2 & y & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & y \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 5 & y & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 5 & y \\ 0 & 0 & 0 & 8 \end{bmatrix},$$

but

$$I_2 \otimes G_3 = \begin{bmatrix} 5 & y & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 5 & y \\ 0 & 0 & 0 & 8 \end{bmatrix},$$

hence

$$Tor_2^A(M, N) = 0.$$

We conclude that

$$Tor_r^A(M, N) = 0 \text{ , for } r \ge 1.$$

It is well known (see [16]) that $Tor_r^A(M,N)$ could be also computed using a free resolution of M. In this case (4.6) should be replaced by

$$Syz[Syz[F_r \otimes I_t]|F_{r+1} \otimes I_t]. \tag{4.7}$$

So, for $r = 1$ we have

$$\ker(F_1 \otimes I_3) = Syz[F_1 \otimes I_3] = Syz \begin{bmatrix} 5y & 0 & 0 \\ 0 & 5y & 0 \\ 0 & 0 & 5y \\ 5x & 0 & 0 \\ 0 & 5x & 0 \\ 0 & 0 & 5x \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

but

$$F_2 \otimes I_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

hence $\ker(F_1 \otimes I_3) = Im(F_2 \otimes I_3)$ and

$$Tor_1^A(M,N) = 0.$$

For $r = 2$ we have

$$\ker(F_2 \otimes I_3) = Syz[F_2 \otimes I_3] = Syz \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

but

$$F_3 \otimes I_3 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

hence

$$Tor_2^A(M,N) = 0.$$

We conclude again that

$$Tor_r^A(M,N) = 0 \text{ , for } r \geq 1.$$

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