

Asymptotic behavior of a reaction-diffusion problem with delay and reaction term concentrated in the boundary

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“Celebrating the 80th birthday of Waldyr Muniz Oliva”

Abstract. In this work we analyze the asymptotic behavior of the solutions of a reaction-diffusion problem with delay when the reaction term is concentrated in a neighborhood of the boundary and this neighborhood shrinks to boundary, as a parameter ϵ goes to zero. This analysis of the asymptotic behavior uses, as a main tool, the convergence result found in [3]. Here, we prove the existence of a family of global attractors and that this family is upper semicontinuous at $\epsilon = 0$. We also prove the continuity of the set of equilibria at $\epsilon = 0$.

1. Introduction

Let Ω be an open bounded set in \mathbb{R}^n with a smooth boundary $\partial\Omega$. We define the strip of width ϵ and base $\partial\Omega$ as

$$\omega_\epsilon = \{x - \sigma \vec{n}(x) : x \in \partial\Omega \quad \text{and} \quad \sigma \in [0, \epsilon)\},$$

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for sufficiently small ϵ , say $0 < \epsilon \leq \epsilon_0$, where $\vec{n}(x)$ denotes the outward normal vector at $x \in \partial\Omega$. We note that the set ω_ϵ has Lebesgue measure $|\omega_\epsilon| = O(\epsilon)$ with $|\omega_\epsilon| \leq k|\partial\Omega|\epsilon$, for some $k > 0$ independent of ϵ , and that for small ϵ , the set ω_ϵ is a neighborhood of $\partial\Omega$ in $\bar{\Omega}$, that collapses to the boundary when the parameter ϵ goes to zero.

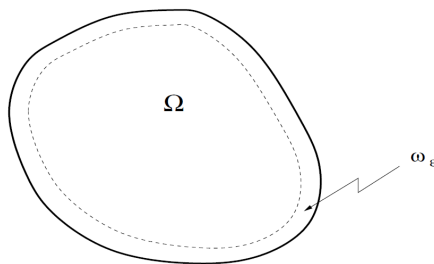


FIGURE 1. The set ω_ϵ .

We are interested in the behavior, for small ϵ , of the solutions of the reaction-diffusion problem with delay in the interior

$$\begin{cases} \frac{\partial u^\epsilon}{\partial t} = \Delta u^\epsilon - \lambda u^\epsilon + \frac{1}{\epsilon} \mathcal{X}_{\omega_\epsilon} f(u^\epsilon(t), u^\epsilon(t - \tau)), & \Omega \times (0, \infty) \\ \frac{\partial u^\epsilon}{\partial n} = 0, & \partial\Omega \times (0, \infty) \\ u^\epsilon = \varphi^\epsilon, & \Omega \times [-\tau, 0]. \end{cases} \quad (1.1)$$

In [3] was proved that, under certain conditions, the limit problem of (1.1) is the following parabolic problem with delay in the boundary

$$\begin{cases} \frac{\partial u^0}{\partial t} = \Delta u^0 - \lambda u^0, & \Omega \times (0, \infty) \\ \frac{\partial u^0}{\partial n} = f(u^0(t), u^0(t - \tau)), & \partial\Omega \times (0, \infty) \\ u^0 = \varphi^0, & \Omega \times [-\tau, 0] \end{cases} \quad (1.2)$$

where $\lambda > 0$, $\tau > 0$ is the delay, $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the nonlinearity, $\varphi^\epsilon : \bar{\Omega} \times [-\tau, 0] \rightarrow \mathbb{R}$, $0 \leq \epsilon \leq \epsilon_0$, is the initial condition and $\mathcal{X}_{\omega_\epsilon}$ is the characteristic function of the set ω_ϵ , $0 < \epsilon \leq \epsilon_0$. Thus, the effective reaction in (1.1) is concentrated in ω_ϵ . Since $\lambda > 0$ in (1.1) and (1.2), then the elliptic problem with homogeneous Neumann boundary conditions, associated to problems (1.1) and (1.2), is positive. In fact, we are supposing $\lambda > 0$ for convenience, since we can consider the problems (1.1) and (1.2) with $\lambda = 0$, add and subtract in the equation the term $\beta u^\epsilon(x, t)$ with $\beta > 0$,

for example $\beta = 1$, and also obtain that the elliptic problem associated is positive.

Here, we will prove the existence of a family of global attractors of (1.1) and (1.2) and that this family is upper semicontinuous at $\epsilon = 0$. We will study the simplest elements from the attractor, the equilibrium solutions. We will show the continuity of the family of equilibria of (1.1) and (1.2) at $\epsilon = 0$.

This kind of problem was initially studied in [6], where linear elliptic equations with terms concentrated were considered and convergence results of the solutions were proved. Later, the asymptotic behavior of the attractors of a parabolic problem without delay was analyzed in [11], where the upper semicontinuity of attractors at $\epsilon = 0$ was proved. The same technique of [6] has been used in [3], where some results of [6] were extended to a reaction-diffusion problem with delay. Thus, our goal is to extend the results of [11] to parabolic problems with delay.

In order to prove the results of this paper, besides of the hypotheses (H1) and (H2) previously assumed in [3] and given by:

(H1) $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is locally Lipschitz.

(H2) $f(0, v) \geq 0$, for all $v \in \mathbb{R}$, and there exist $D \in \mathbb{R}$ and $E \geq 0$ such that

$$f(u, v) \leq Du + E, \quad \forall u, v \geq 0.$$

We will need of the following additional hypotheses:

(H3) $D \in \mathbb{R}$ is such that the first eigenvalue, λ_1^0 , of the following problem is positive

$$\begin{cases} -\Delta\psi_i^0 + \lambda\psi_i^0 = \lambda_i^0\psi_i^0, & \Omega \\ \frac{\partial\psi_i^0}{\partial n} = D\psi_i^0, & \partial\Omega \end{cases}$$

with $i = 1, 2, \dots$

(H4) $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a $C^2(\mathbb{R}^2)$ -function.

Interesting applications of the problems (1.1) and (1.2) with $\lambda \geq 0$, appear in logistic type equations, for example, when the nonlinearity is $f(u, v) = u(1 - v)$ or $f(u, v) = u(a - bu - cv)$, with $a, b, c > 0$ and $u, v \in \mathbb{R}$. The function $f(u, v) = u(a - bu - cv)$ satisfies the hypotheses (H1), (H2), (H3) and (H4), thus the results about the upper semicontinuity of attractor and continuity of equilibria at $\epsilon = 0$ hold in this case. In the case of the function $f(u, v) = u(1 - v)$, the hypotheses (H1), (H2) and (H4) are satisfied, however (H3) is only satisfied for some values of $\lambda > 0$. In this case, we do not know if the results about the upper semicontinuity of

attractors and the continuity of equilibria at $\epsilon = 0$ hold for $\lambda = 0$, since for $\lambda = 0$ in (1.1) and (1.2), the hypothesis (H3) is not satisfied.

The paper will proceed as follows: in Section 2, assuming the hypotheses (H1) and (H2), we will give the notation that it will be used in this paper and we will remember some results obtained in [3]. We will be interested only in nonnegative solutions of (1.1) and (1.2), thus we will consider only nonnegative initial conditions of (1.1) and (1.2), since this will implicate the positiveness of the solutions, as we already saw in [3]. In Section 3, besides of the hypotheses (H1) and (H2), we will need of the dissipative condition (H3) to show that the nonlinear semigroup, associated to the solutions of (1.1) and (1.2), is uniformly bounded for all time, asymptotically smooth and dissipative. With this, we will show the existence of a family of attractors of (1.1) and (1.2) and that this family is upper semicontinuous at $\epsilon = 0$. In particular, we will get the upper semicontinuity of the set of equilibria of (1.1) and (1.2) at $\epsilon = 0$. Afterwards, in Section 4, besides of the hypotheses (H2) and (H3), we will also use the hypothesis (H4) to obtain some technical lemmas. With this and with the results of Section 3, we will prove the lower semicontinuity of the set of equilibria at $\epsilon = 0$ and so the continuity. For so much, we will also need to assume that the equilibrium points of (1.2) are stable under perturbation. This stability under perturbation can be given excluding the zero of spectrum or by the hyperbolicity of equilibrium point.

2. Notation and previous results

Initially, let us denote by $H_p^s(\Omega)$ the Bessel Potential spaces of order $s \in \mathbb{R}$ in an arbitrary domain $\Omega \subset \mathbb{R}^n$, with $1 < p < \infty$ and $H_p^0(\Omega) = L^p(\Omega)$.

We consider the linear operator $A : D(A) \subset L^p(\Omega) \longrightarrow L^p(\Omega)$ given by $Au = -\Delta u + \lambda u$, for all $u \in D(A)$, with domain $D(A) = \{u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$. The operator A is closed densely defined and sectorial in $L^p(\Omega)$, with compact resolvent set $\rho(A)$. Since $\lambda > 0$ then A is a positive operator.

Following [12], where the definition of fractional power was extended to include negative powers and the operator A was extended, using the results of [14] and the interpolation-extrapolation techniques of [1, 2], we know that the operator A has an associated scale of Banach spaces X^β , $\beta \in \mathbb{R}$. Moreover, the operator A , or more properly speaking, the realization of the operator A in X^β , is a sectorial operator in X^β with domain $X^{1+\beta}$.

Let us denote the realization of operator A in the extrapolated spaces $X^{-\beta}$, $0 < \beta < 1$, by $A_{-\beta}$. It follows from results of [10] that the operator $-A_{-\beta}$ generates an analytic semigroup $\{e^{-A_{-\beta}t} : t \geq 0\}$ in $X^{-\beta}$, for

$0 < \beta < 1$, which satisfies, for $-\beta < \alpha < 1 - \beta$,

$$\begin{aligned} \|e^{-A-\beta t}v\|_{X^\alpha} &\leq M e^{-\delta t} \|v\|_{X^\alpha}, & t \geq 0 \\ \|e^{-A-\beta t}v\|_{X^\alpha} &\leq M e^{-\delta t} t^{-(\alpha+\beta)} \|v\|_{X^{-\beta}}, & t > 0, \end{aligned} \tag{2.3}$$

for some $\delta > 0$ and $M > 0$.

We want to choose α , β and p in such a way that

- (1) $X^\alpha \hookrightarrow C(\bar{\Omega})$;
- (2) $X^{1-\beta} = H_p^{2(1-\beta)}(\Omega)$, in other words, $X^{1-\beta}$ does not incorporate the boundary condition;
- (3) $0 < \alpha + \beta < 1$.

So we take α , β and p satisfying

$$\frac{n}{2p} < \alpha < 1 - \beta < 1 - \frac{1}{2p'} = \frac{1}{2} + \frac{1}{2p}. \tag{2.4}$$

Proposition 2.1. *If α , β and p satisfy (2.4), then*

$$X^\alpha = H_p^{2\alpha}(\Omega) \quad \text{and} \quad X^{-\beta} = (H_{p'}^{2\beta}(\Omega))'.$$

Since our equations have time delays, we also need of the following notation:

Notation 2.2. *For a given $\alpha \in \mathbb{R}$, we denote by $C_\alpha = C([- \tau, 0], X^\alpha)$ the Banach space of all continuous functions $u : [- \tau, 0] \rightarrow X^\alpha$ with the norm*

$$\|u\|_{C_\alpha} = \sup_{\theta \in [- \tau, 0]} \|u(\theta)\|_{X^\alpha}.$$

Since we plan to use the linear operator A with homogeneous boundary conditions, to define the abstract problems associated to (1.1) and (1.2), we need to include the nonlinear boundary conditions in the equation. This is done as follows.

Notation 2.3. *Denote by*

$$\begin{cases} F_0 : C_\alpha \longrightarrow X^{-\beta} \\ u \longmapsto F_0(u) = (f_0)_\gamma(u(0), u(-\tau)) \end{cases}$$

and

$$\begin{cases} F_\epsilon : C_\alpha \longrightarrow X^{-\beta} & 0 < \epsilon \leq \epsilon_0 \\ u \longmapsto F_\epsilon(u) = (f_\epsilon)_\Omega(u(0), u(-\tau)) \end{cases}$$

where $(f_0)_\gamma, (f_\epsilon)_\Omega : X^\alpha \times X^\alpha \rightarrow X^{-\beta}$, $0 < \epsilon \leq \epsilon_0$, denote the maps defined, respectively, by

$$\langle (f_0)_\gamma(u, v), \phi \rangle := \int_{\partial\Omega} \gamma(f(u(x), v(x))) \gamma(\phi(x)) dx,$$

$\forall (u, v) \in X^\alpha \times X^\alpha$ and $\forall \phi \in H_{p'}^{2\beta}(\Omega)$, where γ denotes the trace operator, and

$$\langle (f_\epsilon)_\Omega(u, v), \phi \rangle := \frac{1}{\epsilon} \int_{\omega_\epsilon} f(u(x), v(x)) \phi(x) dx$$

$\forall (u, v) \in X^\alpha \times X^\alpha$ and $\forall \phi \in H_{p'}^{2\beta}(\Omega)$.

Thus the problems (1.1) and (1.2) will take the abstract form

$$\begin{cases} \dot{u}^\epsilon(t) + A_{-\beta} u^\epsilon(t) = F_\epsilon(u_t^\epsilon), & t > 0 \quad \text{and} \quad 0 \leq \epsilon \leq \epsilon_0 \\ u^\epsilon(t) = \varphi^\epsilon(t), & t \in [-\tau, 0] \end{cases} \quad (2.5)$$

where, for each $0 \leq \epsilon \leq \epsilon_0$, $u_t^\epsilon : [-\tau, 0] \rightarrow X^\alpha$ denotes the function $u_t^\epsilon(\theta) = u^\epsilon(t + \theta)$, $\theta \in [-\tau, 0]$.

The hypothesis (H1) and the condition (2.4) imply that F_ϵ is locally Lipschitz, uniformly in ϵ . Hence, we have local existence and uniqueness of the solutions of (2.5) or of (1.1) and (1.2) in the weak sense. In [3, Theorem 12], using abstract results of comparison of [5], assuming also the hypothesis (H2) and considering only nonnegative initial conditions of (1.1) and (1.2), in the sense that $\varphi^\epsilon(t) \geq 0$, for all $t \in [-\tau, 0]$, where \leq is the order relationship in $L^p(\Omega)$, was proved that the solutions of our problems with delay (1.1) and (1.2) are nonnegative, that is, for each $0 \leq \epsilon \leq \epsilon_0$, $u^\epsilon(x, t) \geq 0$, for all $(x, t) \in \bar{\Omega} \times [-\tau, \infty)$. Moreover, for each $0 \leq \epsilon \leq \epsilon_0$, $u^\epsilon(x, t) \leq v^\epsilon(x, t)$, for all $(x, t) \in \bar{\Omega} \times [0, \infty)$, where v^ϵ is the weak solution of the following linear parabolic problems without delay

$$\begin{cases} \frac{\partial v^\epsilon}{\partial t} = \Delta v^\epsilon - \lambda v^\epsilon + \frac{1}{\epsilon} \mathcal{X}_{\omega_\epsilon}(Dv^\epsilon + E), & \Omega \times (0, \infty) \\ \frac{\partial v^\epsilon}{\partial n} = 0, & \partial\Omega \times (0, \infty) \\ v^\epsilon(0) = \varphi^\epsilon(0), & \Omega \end{cases} \quad (2.6)$$

$$\begin{cases} \frac{\partial v^0}{\partial t} = \Delta v^0 - \lambda v^0, & \Omega \times (0, \infty) \\ \frac{\partial v^0}{\partial n} = Dv^0 + E, & \partial\Omega \times (0, \infty) \\ v^0(0) = \varphi^0(0), & \Omega. \end{cases} \quad (2.7)$$

The global existence, uniqueness and continuous dependence of the solutions of (2.6) and (2.7) follow of [10] and, for each $0 \leq \epsilon \leq \epsilon_0$, we have well

defined semigroups in X^α ,

$$\begin{aligned} T^\epsilon(t) : X^\alpha &\longrightarrow X^\alpha \\ \varphi &\longmapsto T^\epsilon(t)\varphi = v^\epsilon(t; \varphi), \quad t \geq 0. \end{aligned}$$

Using comparison, in [3] was showed that the solutions of (1.1) and (1.2) are globally defined. Thus, for each $0 \leq \epsilon \leq \epsilon_0$, we have well defined semigroups in C_α ,

$$\begin{aligned} U^\epsilon(t) : C_\alpha &\longrightarrow C_\alpha \\ \varphi &\longmapsto U^\epsilon(t)\varphi = u_t^\epsilon(\varphi), \quad t \geq 0. \end{aligned}$$

We note that $u^\epsilon : [-\tau, \infty) \longrightarrow X^\alpha$ satisfies the variation of constants formula

$$u^\epsilon(t) = \begin{cases} e^{-A-\beta t}\varphi(0) + \int_0^t e^{-A-\beta(t-s)} F_\epsilon(u_s^\epsilon) ds, & t \geq 0 \\ \varphi(t), & -\tau \leq t \leq 0. \end{cases} \quad (2.8)$$

Moreover, in [3] was proved that the solutions of (1.1) and (1.2), with uniformly bounded initial conditions in C_α , are also uniformly bounded in C_α , for t in finite and positive time intervals. This uniform boundedness of the solutions was necessary to prove the main result of [3], namely, convergence theorem of the solutions and given by:

Theorem 2.4. *Suppose that (H1) and (H2) hold and that α , β and p satisfy (2.4). Let $0 < T < \infty$ and $B \subset C_\alpha$ be a bounded set. For each $0 \leq \epsilon \leq \epsilon_0$, let $\varphi^\epsilon \in C_\alpha$ such that $\varphi^\epsilon \rightarrow \varphi^0$ in C_α , as $\epsilon \rightarrow 0$, with $\varphi^0 \in B$. Then, there exist $M(T, B) > 0$ and $M(\epsilon) \geq 0$, with $M(\epsilon) \rightarrow 0$, as $\epsilon \rightarrow 0$, such that*

$$\|U^\epsilon(t)\varphi^\epsilon - U^0(t)\varphi^0\|_{C_\alpha} \leq M(T, B)M(\epsilon), \quad \forall t \in (0, T].$$

So $U^\epsilon(t)\varphi^\epsilon \rightarrow U^0(t)\varphi^0$ in C_α , as $\epsilon \rightarrow 0$, uniformly for $\varphi^0 \in B$ and $t \in [0, T]$.

3. Existence and upper semicontinuity of attractors

In this section we will prove the existence of a family of the global attractors of (1.1) and (1.2) and that this family is upper semicontinuous at $\epsilon = 0$. In particular, we will get that the set of equilibria of (1.1) and (1.2) is upper semicontinuous at $\epsilon = 0$. As in [3], we compare the solutions of (1.1) and (1.2) with the solutions of the linear parabolic problems without delay (2.6) and (2.7). The advantage of this comparison is that the asymptotic behavior of the attractors of (2.6) and (2.7) was already studied in [11], using the results of [6], on concentrating integrals and elliptic problem associated to parabolic problem, and some previous results of [5]. Some results obtained in [11] and adapted to the case where the space is X^α , with $X^\alpha \hookrightarrow C(\bar{\Omega})$, are given in the following theorem:

Theorem 3.1. *Suppose that $D \in \mathbb{R}$ and $E \geq 0$ in (2.6) and (2.7), that (H3) holds and that α, β and p satisfy (2.4). Then, there exists $\epsilon_0 > 0$ such that:*

- (1) *If B is a bounded subset of X^α , then, for each $0 \leq \epsilon \leq \epsilon_0$, $\bigcup_{t \geq 0} T^\epsilon(t)B$ is a bounded subset of X^α , uniformly in ϵ .*
- (2) *The problems (2.6) and (2.7) have a global attractor \mathcal{B}_ϵ in X^α , for each $0 \leq \epsilon \leq \epsilon_0$.*
- (3) *There exists $K > 0$ independent of ϵ such that*

$$\sup_{\epsilon \in [0, \epsilon_0]} \sup_{v \in \mathcal{B}_\epsilon} \|v\|_{X^\alpha} \leq K.$$

As a consequence of Theorem 3.1, we obtain:

Corollary 3.2. *Suppose that $D \in \mathbb{R}$ and $E \geq 0$ in (2.6) and (2.7), that (H3) holds and that α, β and p satisfy (2.4). Let $\epsilon_0 > 0$ be as in the Theorem 3.1 and, for each $0 \leq \epsilon \leq \epsilon_0$ and $\eta > 0$, let V_η^ϵ be a η -neighborhood of \mathcal{B}_ϵ , that is,*

$$V_\eta^\epsilon = \{v \in X^\alpha : \text{dist}(v, \mathcal{B}_\epsilon) := \inf_{v_\epsilon \in \mathcal{B}_\epsilon} \|v - v_\epsilon\|_{X^\alpha} < \eta\}. \quad (3.9)$$

Then, V_η^ϵ is an absorbing set in X^α for $T^\epsilon(t)$.

So, by comparison, we will show that the solutions of (1.1) and (1.2), with bounded initial conditions in C_α , are uniformly bounded in C_α for all time and, consequently, the existence and upper semicontinuity of attractors.

3.1. Existence of global attractors. We will show that, for each $0 \leq \epsilon \leq \epsilon_0$, the semigroup associated to the solutions of (1.1) and (1.2), $\{U^\epsilon(t) : t \geq 0\}$, has a global attractor. Using the results of [8], it is enough show that $\{U^\epsilon(t) : t \geq 0\}$ is asymptotically smooth, point dissipative and that orbits of bounded set of C_α are bounded in C_α . Initially, we have the following result:

Proposition 3.3. *Suppose that (H1) holds and that α, β and p satisfy (2.4). Then, for each $0 \leq \epsilon \leq \epsilon_0$, the semigroup $U^\epsilon(t)$ is conditionally completely continuous for each $t > \tau$ fixed. Thus, $U^\epsilon(t)$ is asymptotically smooth.*

Proof. The proof follows of [13, Proposition 2.4] and [8, Corollary 3.2.2]. \square

Now, we will see that orbits of bounded set of C_α are bounded in C_α , uniformly in ϵ . We will abuse of the notation and only write C_α instead of $C_\alpha^+ = C([-\tau, 0], H_p^{2\alpha}(\Omega)^+)$, with $H_p^{2\alpha}(\Omega)^+ = \{f \in H_p^{2\alpha}(\Omega) : f \geq 0\}$, where \leq is the order relationship in $L^p(\Omega)$.

Lemma 3.4. *Suppose that (H1), (H2) and (H3) hold, that α, β and p satisfy (2.4) and let $\epsilon_0 > 0$ be as in the Theorem 3.1. If B is a bounded set of C_α , then, for each $0 \leq \epsilon \leq \epsilon_0$, $\bigcup_{t \geq 0} U^\epsilon(t)B$ is a bounded set of C_α , uniformly in ϵ .*

Proof. Taking any $\varphi \in B$, we have that there exists $R = R(B) > 0$ such that $\|\varphi\|_{C_\alpha} \leq R$. Thus, using the item 1 of Theorem 3.1 and $X^\alpha \hookrightarrow C(\bar{\Omega})$, there exists $K_1 = K_1(B) > 0$ independent of ϵ such that

$$\|v^\epsilon(t)\|_{C(\bar{\Omega})} = \|T^\epsilon(t)\varphi(0)\|_{C(\bar{\Omega})} \leq K_1, \quad \forall t \geq 0 \quad \text{and} \quad \forall 0 \leq \epsilon \leq \epsilon_0.$$

Now, from [3, Theorem 12], for each $0 \leq \epsilon \leq \epsilon_0$, $u^\epsilon(x, t) \geq 0$, for all $(x, t) \in \bar{\Omega} \times [-\tau, \infty)$, and

$$u^\epsilon(x, t) \leq v^\epsilon(x, t), \quad \forall (x, t) \in \bar{\Omega} \times [0, \infty).$$

Therefore, there exists $\bar{K} = \bar{K}(B) > 0$ independent of ϵ such that

$$\|u^\epsilon(t + \theta)\|_{C(\bar{\Omega})} \leq \bar{K}, \quad \forall t \geq 0 \quad \forall \theta \in [-\tau, 0] \quad \text{and} \quad \forall 0 \leq \epsilon \leq \epsilon_0. \quad (3.10)$$

For $t > 0$ and $0 \leq \epsilon \leq \epsilon_0$, using (2.8) and the estimatives in (2.3), we have

$$\begin{aligned} \|(U^\epsilon(t)\varphi)(0)\|_{X^\alpha} &= \|u^\epsilon(t)\|_{X^\alpha} \\ &\leq MR + M \int_0^t e^{-\delta(t-s)}(t-s)^{-(\alpha+\beta)} \|F_\epsilon(u_s^\epsilon)\|_{X^{-\beta}} ds. \end{aligned}$$

For $0 < \epsilon \leq \epsilon_0$ and $0 \leq s \leq t$, applying the Hölder's Inequality, we have

$$|\langle F_\epsilon(u_s^\epsilon), \phi \rangle| \leq \left(\frac{1}{\epsilon} \int_{\omega_\epsilon} |f(u^\epsilon(x, s), u^\epsilon(x, s - \tau))|^p dx \right)^{\frac{1}{p}} \left(\frac{1}{\epsilon} \int_{\omega_\epsilon} |\phi(x)|^{p'} dx \right)^{\frac{1}{p'}}.$$

Now, from [6, Lemma 2.1] we have that there exists a constant $C > 0$ independent of ϵ such that

$$\left(\frac{1}{\epsilon} \int_{\omega_\epsilon} |\phi(x)|^{p'} dx \right)^{\frac{1}{p'}} \leq C \|\phi\|_{H_{p'}^{2\beta}(\Omega)}, \quad \forall \phi \in H_{p'}^{2\beta}(\Omega). \quad (3.11)$$

Using (3.10) and (3.11), we get

$$\begin{aligned} \|F_\epsilon(u_s^\epsilon)\|_{X^{-\beta}} &\leq C \left(\frac{1}{\epsilon} \int_{\omega_\epsilon} |f(u^\epsilon(x, s), u^\epsilon(x, s - \tau))|^p dx \right)^{\frac{1}{p}} \\ &\leq C \left(\frac{|\omega_\epsilon|}{\epsilon} \right)^{\frac{1}{p}} \sup_{|u|, |v| \leq \bar{K}} |f(u, v)| \leq C_1, \end{aligned}$$

where $C_1 = C_1(B)$ does not depend of ϵ , we note that $|\omega_\epsilon| \leq k|\partial\Omega|\epsilon$, for some $k > 0$ independent of ϵ .

For $\epsilon = 0$ and $0 \leq s \leq t$, applying the Hölder's Inequality, we have

$$\begin{aligned} & |\langle F_0(u_s^0), \phi \rangle| \leq \\ & \leq \left(\int_{\partial\Omega} |\gamma(f(u^0(x, s), u^0(x, s - \tau)))|^p dx \right)^{\frac{1}{p}} \left(\int_{\partial\Omega} |\gamma(\phi(x))|^{p'} dx \right)^{\frac{1}{p'}} \\ & \leq \bar{C} \left(\int_{\partial\Omega} |\gamma(f(u^0(x, s), u^0(x, s - \tau)))|^p dx \right)^{\frac{1}{p}} \|\phi\|_{H_{p'}^{2\beta}(\Omega)}, \quad \forall \phi \in H_{p'}^{2\beta}(\Omega), \end{aligned}$$

where, in the last passage, we used the continuity of trace operator $\gamma : H_{p'}^{2\beta}(\Omega) \rightarrow L^{p'}(\partial\Omega)$. Using (3.10) we get

$$\|F_0(u_s^0)\|_{X^{-\beta}} \leq \bar{C} |\partial\Omega|^{\frac{1}{p}} \sup_{|u|, |v| \leq \bar{K}} |f(u, v)| \leq C_2.$$

Therefore, there exists $\bar{M} = \bar{M}(B) > 0$ independent of ϵ such that

$$\|u^\epsilon(t)\|_{X^\alpha} \leq MR + \bar{M} \int_0^t e^{-\delta(t-s)} (t-s)^{-(\alpha+\beta)} ds, \quad \text{for } t > 0 \text{ and } 0 \leq \epsilon \leq \epsilon_0.$$

Given any $\theta \in [-\tau, 0]$, if $t + \theta > 0$ then

$$\begin{aligned} \|u^\epsilon(t + \theta)\|_{X^\alpha} & \leq MR + \bar{M} \int_0^{t+\theta} e^{-\delta(t+\theta-s)} (t+\theta-s)^{-(\alpha+\beta)} ds \\ & \leq MR + \bar{M} \int_0^t e^{-\delta s} s^{-(\alpha+\beta)} ds. \end{aligned}$$

Taking the supreme in $\theta \in [-\tau, 0]$, we get

$$\begin{aligned} \|u_t^\epsilon\|_{C_\alpha} & \leq MR + \bar{M} \int_0^\infty e^{-\delta s} s^{-(\alpha+\beta)} ds \\ & = MR + \bar{M} \delta^{(\alpha+\beta)-1} \Gamma(1 - (\alpha + \beta)) = K_0, \quad \forall 0 \leq \epsilon \leq \epsilon_0, \end{aligned}$$

where

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \quad \Re(z) > 0 \quad (\text{Gamma Function}).$$

If $-\tau \leq t + \theta \leq 0$ and $t > 0$, then

$$\|u^\epsilon(t + \theta)\|_{X^\alpha} = \|\varphi(t + \theta)\|_{X^\alpha} \leq \|\varphi\|_{C_\alpha} \leq R.$$

Taking again the supreme in $\theta \in [-\tau, 0]$, we get $\|u_t^\epsilon\|_{C_\alpha} \leq R$, for all $0 \leq \epsilon \leq \epsilon_0$.

Therefore, there exists $K_0 = K_0(B) > 0$ such that

$$\|U^\epsilon(t)\varphi\|_{C_\alpha} = \|u_t^\epsilon(\varphi)\|_{C_\alpha} \leq K_0, \quad \forall t \geq 0 \quad \text{and} \quad \forall 0 \leq \epsilon \leq \epsilon_0.$$

We note that the constant K_0 depends of B , but does not depend of ϵ . \square

Before we show that $\{U^\epsilon(t) : t \geq 0\}$ is point dissipative and, consequently, the existence of attractors, we will build an absorbing set in C_α for $U^\epsilon(t)$, for all $0 \leq \epsilon \leq \epsilon_0$. The existence of this absorbing set will be important, mainly, to we show uniform bounded of attractors, which is necessary to prove the upper semicontinuity of attractors.

Remark 3.5. *From item 3 of Theorem 3.1, there exists $K > 0$ independent of ϵ such that $\mathcal{B}_\epsilon \subset \bar{B}_K(0)$, for all $0 \leq \epsilon \leq \epsilon_0$, where $\bar{B}_K(0)$ is the closed ball in X^α with center in the origin and ray K .*

Taking the closed ball $\bar{B}_{2K}(0) \subset X^\alpha$, there exists $\eta > 0$ such that $V_\eta^\epsilon \subset \bar{B}_{2K}(0)$, for all $0 \leq \epsilon \leq \epsilon_0$, where V_η^ϵ is given by (3.9). Now, we define

$$\Sigma = \{u \in C_\alpha : 0 \leq u(\theta)(x) := u(x, \theta) \leq 2cK, \\ \forall \theta \in [-\tau, 0] \text{ and } \forall x \in \bar{\Omega}\}, \tag{3.12}$$

where $c > 0$ is the constant of the continuous embedding $X^\alpha \hookrightarrow C(\bar{\Omega})$. We will prove that Σ is an absorbing set in C_α for $U^\epsilon(t)$, for all $0 \leq \epsilon \leq \epsilon_0$.

Lemma 3.6. *Suppose that (H1), (H2) and (H3) hold, that α , β and p satisfy (2.4) and let $\epsilon_0 > 0$ be as in the Theorem 3.1. Then, Σ is an absorbing set in C_α for $U^\epsilon(t)$, for all $0 \leq \epsilon \leq \epsilon_0$, where Σ is given by (3.12).*

Proof. Let $B \subset C_\alpha$ be a bounded set and $\varphi \in B$. From Remark 3.5 there exists $\eta > 0$ such that $V_\eta^\epsilon \subset \bar{B}_{2K}(0)$, for all $0 \leq \epsilon \leq \epsilon_0$, where V_η^ϵ is given by (3.9).

From Corollary 3.2 there exists $t_0^\epsilon = t_0^\epsilon(\epsilon, \eta, B) > 0$ such that $T^\epsilon(t)\varphi(0) \in V_\eta^\epsilon$, for all $t \geq t_0^\epsilon$. Since $X^\alpha \hookrightarrow C(\bar{\Omega})$ then

$$\|T^\epsilon(t)\varphi(0)\|_{C(\bar{\Omega})} \leq c \|T^\epsilon(t)\varphi(0)\|_{X^\alpha} \leq 2cK, \quad \forall t \geq t_0^\epsilon \quad \text{and} \quad \forall 0 \leq \epsilon \leq \epsilon_0.$$

Using [3, Theorem 12], for $t \geq t_0^\epsilon + \tau$ we get

$$0 \leq (U^\epsilon(t)\varphi)(\theta)(x) \leq (T^\epsilon(t + \theta)\varphi(0))(x) \leq 2cK,$$

$\forall \theta \in [-\tau, 0]$ and $\forall x \in \bar{\Omega}$. Hence, $U^\epsilon(t)\varphi \in \Sigma$, for all $t \geq t_0^\epsilon + \tau$ and for all $\varphi \in B$. Therefore,

$$U^\epsilon(t)B \subset \Sigma, \quad \forall t \geq t_0^\epsilon + \tau \quad \text{and} \quad \forall 0 \leq \epsilon \leq \epsilon_0. \quad \square$$

Lemma 3.7. *Suppose that (H1), (H2) and (H3) hold, that α , β and p satisfy (2.4) and let $\epsilon_0 > 0$ be as in the Theorem 3.1. Then, for each $0 \leq \epsilon \leq \epsilon_0$, $U^\epsilon(t)$ is bounded dissipative in C_α . In particular, for each $0 \leq \epsilon \leq \epsilon_0$, $U^\epsilon(t)$ is point dissipative in C_α .*

Proof. Let $B \subset C_\alpha$ be a bounded set. From Lemma 3.6 there exists $t_0^\epsilon = t_0^\epsilon(\epsilon, B) > 0$ such that $U^\epsilon(t)B \subset \Sigma$, for all $t \geq t_0^\epsilon$. For any $\varphi \in B$ and $t > 0$, similarly to Lemma 3.4, we have

$$\begin{aligned} \|(U^\epsilon(t)(U^\epsilon(t_0^\epsilon)\varphi))(0)\|_{X^\alpha} &\leq Me^{-\delta t} \|U^\epsilon(t_0^\epsilon)\varphi\|_{C_\alpha} + \\ &+ M \int_0^t e^{-\delta(t-s)}(t-s)^{-(\alpha+\beta)} \|F_\epsilon(U^\epsilon(s)(U^\epsilon(t_0^\epsilon)\varphi))\|_{X^{-\beta}} ds. \end{aligned}$$

By Lemma 3.4 there exists $K_0 = K_0(B) > 0$ independent of ϵ such that $\|U^\epsilon(t_0^\epsilon)\varphi\|_{C_\alpha} \leq K_0$. Applying again the Lemma 3.4, there exists $\bar{K}_0 = \bar{K}_0(B) > 0$ independent of ϵ such that

$$\|U^\epsilon(s)(U^\epsilon(t_0^\epsilon)\varphi)\|_{C_\alpha} \leq \bar{K}_0, \quad \forall s \geq 0 \quad \text{and} \quad \forall 0 \leq \epsilon \leq \epsilon_0.$$

Now, using [3, Lemma 9] we have that there exists $\bar{C} = \bar{C}(B) > 0$ independent of ϵ such that

$$\|F_\epsilon(U^\epsilon(s)(U^\epsilon(t_0^\epsilon)\varphi))\|_{X^{-\beta}} \leq \bar{C}, \quad \forall 0 \leq s \leq t \quad \text{and} \quad \forall 0 \leq \epsilon \leq \epsilon_0.$$

Thus, there exists $\bar{M} = \bar{M}(B) > 0$ independent of ϵ such that

$$\|(U^\epsilon(t)(U^\epsilon(t_0^\epsilon)\varphi))(0)\|_{X^\alpha} \leq MK_0 + \bar{M} \int_0^t e^{-\delta(t-s)}(t-s)^{-(\alpha+\beta)} ds,$$

for $t > 0$ and $0 \leq \epsilon \leq \epsilon_0$

Given any $\theta \in [-\tau, 0]$, if $t + \theta > 0$ then

$$\begin{aligned} \|(U^\epsilon(t)(U^\epsilon(t_0^\epsilon)\varphi))(\theta)\|_{X^\alpha} &\leq MK_0 + \bar{M} \int_0^{t+\theta} e^{-\delta(t+\theta-s)}(t+\theta-s)^{-(\alpha+\beta)} ds \\ &\leq MK_0 + \bar{M} \int_0^t e^{-\delta s} s^{-(\alpha+\beta)} ds. \end{aligned}$$

Taking the supreme in $\theta \in [-\tau, 0]$, we get

$$\|U^\epsilon(t+t_0^\epsilon)\varphi\|_{C_\alpha} \leq MK_0 + \bar{M} \int_0^t e^{-\delta s} s^{-(\alpha+\beta)} ds. \quad (3.13)$$

We note that t_0^ϵ and the right side of (3.13) do not depend on $\varphi \in B$. Then, doing $t \rightarrow \infty$ in (3.13), we have

$$\limsup_{t \rightarrow \infty} \|U^\epsilon(t+t_0^\epsilon)\varphi\|_{C_\alpha} \leq MK_0 + \bar{M} \int_0^\infty e^{-\delta s} s^{-(\alpha+\beta)} ds, \quad (3.14)$$

$\forall 0 \leq \epsilon \leq \epsilon_0$.

Since the right side of (3.14) does not depend on $\varphi \in B$, then the set in C_α , which is bounded by right side of (3.14), attracts each bounded set of C_α

through $U^\epsilon(t)$. Therefore, for each $0 \leq \epsilon \leq \epsilon_0$, $U^\epsilon(t)$ is bounded dissipative in C_α . In particular, for each $0 \leq \epsilon \leq \epsilon_0$, $U^\epsilon(t)$ is point dissipative in C_α . \square

Theorem 3.8. *Suppose that (H1), (H2) and (H3) hold, that α , β and p satisfy (2.4) and let $\epsilon_0 > 0$ be as in the Theorem 3.1. Then, the problems (1.1) and (1.2) have a global attractor \mathcal{A}_ϵ in C_α , for each $0 \leq \epsilon \leq \epsilon_0$. Moreover, $\mathcal{A}_\epsilon \subset \Sigma$, for all $0 \leq \epsilon \leq \epsilon_0$, where Σ is given by (3.12).*

Proof. Since, for each $0 \leq \epsilon \leq \epsilon_0$, $U^\epsilon(t)$ is asymptotically smooth (Proposition 3.3), point dissipative (Lemma 3.7) and orbits of bounded set of C_α are bounded in C_α (Lemma 3.4), then by [8], $U^\epsilon(t)$ has a global attractor \mathcal{A}_ϵ .

From Lemma 3.6 there exists $t_0^\epsilon = t_0^\epsilon(\epsilon, \mathcal{A}_\epsilon) > 0$ such that $U^\epsilon(t)\mathcal{A}_\epsilon \subset \Sigma$, for all $t \geq t_0^\epsilon$. For each $0 \leq \epsilon \leq \epsilon_0$, \mathcal{A}_ϵ is invariant for $U^\epsilon(t)$, that is, $\mathcal{A}_\epsilon = U^\epsilon(t)\mathcal{A}_\epsilon$, for all $t \geq 0$. Hence, $\mathcal{A}_\epsilon \subset \Sigma$, for all $0 \leq \epsilon \leq \epsilon_0$. \square

3.2. Upper semicontinuity of attractors and of the set of equilibria. We will show that the family of the global attractors $\{\mathcal{A}_\epsilon\}_{\epsilon \in [0, \epsilon_0]}$ of (1.1) and (1.2) is upper semicontinuous at $\epsilon = 0$. In particular, we will get the upper semicontinuity of the set of equilibria of (1.1) and (1.2) at $\epsilon = 0$. For this, we need of the following lemmas:

Lemma 3.9. *Suppose that (H1), (H2) and (H3) hold, that α , β and p satisfy (2.4) and let $\epsilon_0 > 0$ be as in the Theorem 3.1. Then, there exists $R > 0$ independent of ϵ such that*

$$\sup_{\epsilon \in [0, \epsilon_0]} \sup_{u \in \mathcal{A}_\epsilon} \|u\|_{C_\alpha} \leq R. \tag{3.15}$$

In particular, \mathcal{A}_0 attracts $\bigcup_{\epsilon \in (0, \epsilon_0]} \mathcal{A}_\epsilon$ in C_α .

Proof. From Theorem 3.8 we have that $\mathcal{A}_\epsilon \subset \Sigma$, for all $0 \leq \epsilon \leq \epsilon_0$. We note that this guarantees uniform bounded in ϵ , of attractors in the norm of $C(\bar{\Omega})$. Using this, the invariance of attractors, the variation of constants formula (2.8) and the estimates in (2.3), we can get that there exists $R > 0$ independent of ϵ such that (3.15) holds. Therefore, $\bigcup_{\epsilon \in (0, \epsilon_0]} \mathcal{A}_\epsilon$ is a bounded set in C_α , uniformly in ϵ , with $\bigcup_{\epsilon \in (0, \epsilon_0]} \mathcal{A}_\epsilon \subset \bar{B}_R(0)$, where $\bar{B}_R(0)$ is the closed ball in C_α with center in the origin and ray R . Now, \mathcal{A}_0 attracts $\bar{B}_R(0)$, that is,

$$\text{dist}_{C_\alpha} (U^0(t)\bar{B}_R(0), \mathcal{A}_0) \rightarrow 0, \quad \text{as } t \rightarrow \infty,$$

where

$$\begin{aligned} \text{dist}_{C_\alpha}(U^0(t)\bar{B}_R(0), \mathcal{A}_0) &:= \sup_{v \in \bar{B}_R(0)} \text{dist}(U^0(t)v, \mathcal{A}_0) \\ &= \sup_{v \in \bar{B}_R(0)} \inf_{u_0 \in \mathcal{A}_0} \left\{ \|U^0(t)v - u_0\|_{C_\alpha} \right\}. \end{aligned}$$

Thus, given $\eta > 0$, there exists $t_0 = t_0(\bar{B}_R(0), \eta) > 0$ independent of ϵ such that

$$\text{dist}(U^0(t)v, \mathcal{A}_0) < \eta, \quad \forall t > t_0 \quad \text{and} \quad \forall v \in \bar{B}_R(0).$$

In particular, for all $v \in \bigcup_{\epsilon \in (0, \epsilon_0]} \mathcal{A}_\epsilon \subset \bar{B}_R(0)$ we have

$$\text{dist}(U^0(t)v, \mathcal{A}_0) < \eta, \quad \forall t > t_0.$$

Hence, \mathcal{A}_0 attracts $\bigcup_{\epsilon \in (0, \epsilon_0]} \mathcal{A}_\epsilon$ in C_α . We observe that the attraction time does not depend of ϵ . \square

Lemma 3.10. *Suppose that (H1), (H2) and (H3) hold, that α, β and p satisfy (2.4), let $\epsilon_0 > 0$ be as in the Theorem 3.1 and $0 < T < \infty$. Then, there exist $C(T) > 0$ and $C(\epsilon) \geq 0$, with $C(\epsilon) \rightarrow 0$, as $\epsilon \rightarrow 0$, such that for $\varphi_\epsilon \in \mathcal{A}_\epsilon, 0 < \epsilon \leq \epsilon_0$,*

$$\|U^\epsilon(t)\varphi_\epsilon - U^0(t)\varphi_\epsilon\|_{C_\alpha} \leq C(T)C(\epsilon), \quad \forall t \in (0, T].$$

Proof. The proof follows of the continuous dependence of solutions in relation to the initial data and of the convergence theorem of the nonlinear semigroup given by Theorem 2.4. \square

Theorem 3.11. *Suppose that (H1), (H2) and (H3) hold, that α, β and p satisfy (2.4) and let $\epsilon_0 > 0$ be as in the Theorem 3.1. Then, the family of global attractors of (1.1) and (1.2), $\{\mathcal{A}_\epsilon\}_{\epsilon \in [0, \epsilon_0]}$, is upper semicontinuous at $\epsilon = 0$ in C_α , that is,*

$$\text{dist}_{C_\alpha}(\mathcal{A}_\epsilon, \mathcal{A}_0) \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0,$$

where

$$\text{dist}_{C_\alpha}(\mathcal{A}_\epsilon, \mathcal{A}_0) := \sup_{u_\epsilon \in \mathcal{A}_\epsilon} \text{dist}(u_\epsilon, \mathcal{A}_0) = \sup_{u_\epsilon \in \mathcal{A}_\epsilon} \inf_{u_0 \in \mathcal{A}_0} \left\{ \|u_\epsilon - u_0\|_{C_\alpha} \right\}.$$

Proof. From Lemma 3.9, \mathcal{A}_0 attracts $\bigcup_{\epsilon \in (0, \epsilon_0]} \mathcal{A}_\epsilon$, and we saw in the proof of Lemma 3.9 that given $\eta > 0$, there exists $T = T(\eta) > 0$ independent of ϵ such that

$$\text{dist}(U^0(T)u_\epsilon, \mathcal{A}_0) = \inf_{u_0 \in \mathcal{A}_0} \|U^0(T)u_\epsilon - u_0\|_{C_\alpha} \leq \frac{\eta}{2},$$

$$\forall u_\epsilon \in \mathcal{A}_\epsilon \text{ with } \epsilon \in (0, \epsilon_0].$$

Using that \mathcal{A}_ϵ is invariant, given $v_\epsilon \in \mathcal{A}_\epsilon$, there exists $u_\epsilon \in \mathcal{A}_\epsilon$ such that $U^\epsilon(T)u_\epsilon = v_\epsilon$. Thus,

$$\text{dist}(v_\epsilon, \mathcal{A}_0) = \inf_{u_0 \in \mathcal{A}_0} \|v_\epsilon - u_0\|_{C_\alpha} \leq \|v_\epsilon - U^0(T)u_\epsilon\|_{C_\alpha} + \text{dist}(U^0(T)u_\epsilon, \mathcal{A}_0).$$

From Lemma 3.10, if ϵ is small sufficiently, we get

$$\|v_\epsilon - U^0(T)u_\epsilon\|_{C_\alpha} = \|U^\epsilon(T)u_\epsilon - U^0(T)u_\epsilon\|_{C_\alpha} \leq \frac{\eta}{2}.$$

Therefore, for ϵ small sufficiently,

$$\text{dist}(v_\epsilon, \mathcal{A}_0) \leq \eta, \quad \forall v_\epsilon \in \mathcal{A}_\epsilon.$$

□

In particular, we get the upper semicontinuity of the set of equilibria of (1.1) and (1.2).

Corollary 3.12. *Suppose that (H1), (H2) and (H3) hold and that α , β and p satisfy (2.4). Then, for every sequence $\epsilon \rightarrow 0$ and for every sequence of equilibria $u_\epsilon^* \in \mathcal{A}_\epsilon$ of (1.1) there exist a subsequence, that we still denote by ϵ , and an equilibrium point $u_0^* \in \mathcal{A}_0$ of (1.2) such that*

$$\|u_\epsilon^* - u_0^*\|_{C_\alpha} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

Proof. From the upper semicontinuity of the attractors given by Theorem 3.11, we obtain the existence of a $u_0^* \in \mathcal{A}_0$ such that

$$\|u_\epsilon^* - u_0^*\|_{C_\alpha} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

To show that u_0^* is an equilibrium point of (1.2), we observe that for any $t > 0$,

$$\begin{aligned} \|u_\epsilon^* - U^0(t)u_0^*\|_{C_\alpha} &\leq \|u_\epsilon^* - u_0^*\|_{C_\alpha} + \|u_0^* - U^0(t)u_0^*\|_{C_\alpha} \\ &\rightarrow \|u_0^* - U^0(t)u_0^*\|_{C_\alpha}, \quad \text{as } \epsilon \rightarrow 0. \end{aligned}$$

Moreover, for a fixed $T > 0$ and for any $t \in (0, T)$, using that u_ϵ^* is a stationary state of (1.1) and the Theorem 2.4, we get

$$\|u_\epsilon^* - U^0(t)u_0^*\|_{C_\alpha} = \|U^\epsilon(t)u_\epsilon^* - U^0(t)u_0^*\|_{C_\alpha} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

In particular, we have that for each $t > 0$, $u_0^* = U^0(t)u_0^*$, which implies that u_0^* is an equilibrium of (1.2). □

4. Continuity of the set of equilibria

In this section we will study the simplest elements from the attractor, the equilibrium solutions. The equilibrium solutions of (1.1) and (1.2) are those solutions which are independent of time, $U^\epsilon(t)u_\epsilon = u_\epsilon$, for all $t \geq 0$ and $0 \leq \epsilon \leq \epsilon_0$, that is, the solutions of the respective elliptic problems

$$\begin{cases} -\Delta u_\epsilon + \lambda u_\epsilon = \frac{1}{\epsilon} \mathcal{X}_{\omega_\epsilon} f(u_\epsilon, u_\epsilon), & \Omega \\ \frac{\partial u_\epsilon}{\partial n} = 0, & \partial\Omega \end{cases} \quad (4.16)$$

$$\begin{cases} -\Delta u_0 + \lambda u_0 = 0, & \Omega \\ \frac{\partial u_0}{\partial n} = f(u_0, u_0), & \partial\Omega. \end{cases} \quad (4.17)$$

We will prove the continuity of the set of equilibria of the problems (1.1) and (1.2) at $\epsilon = 0$, using some technique developed in [4, 7]. Since the upper semicontinuity already was proved in the Corollary 3.12, then we just have that to prove the lower semicontinuity.

4.1. Abstract setting and technical results. Initially, we will write the elliptic problems (4.16) and (4.17) in the abstract form.

Definition 4.1. *Define*

$$\begin{aligned} (F_0)_\gamma &: X^\alpha \longrightarrow X^{-\beta} \\ u &\longmapsto (F_0)_\gamma(u) \end{aligned}$$

$$\langle (F_0)_\gamma(u), \phi \rangle := \int_{\partial\Omega} \gamma(f(u(x), u(x))) \gamma(\phi(x)) dx, \quad \forall \phi \in H_{p'}^{2\beta}(\Omega),$$

where γ denotes the trace operator. For each $0 < \epsilon \leq \epsilon_0$, define

$$\begin{aligned} (F_\epsilon)_\Omega &: X^\alpha \longrightarrow X^{-\beta} \\ u &\longmapsto (F_\epsilon)_\Omega(u) \end{aligned}$$

$$\langle (F_\epsilon)_\Omega(u), \phi \rangle := \frac{1}{\epsilon} \int_{\omega_\epsilon} f(u(x), u(x)) \phi(x) dx, \quad \forall \phi \in H_{p'}^{2\beta}(\Omega).$$

Using the hypothesis (H4) we can show that $(F_0)_\gamma$ and $(F_\epsilon)_\Omega$, $0 < \epsilon \leq \epsilon_0$, are well defined. So, the problems (4.16) and (4.17) can be written in the abstract form, respectively, as

$$A_{-\beta} u_\epsilon = (F_\epsilon)_\Omega(u_\epsilon), \quad 0 < \epsilon \leq \epsilon_0 \quad (4.18)$$

$$A_{-\beta} u_0 = (F_0)_\gamma(u_0). \quad (4.19)$$

We denote by \mathcal{E}_ϵ , $\epsilon \in [0, \epsilon_0]$, the set of solutions of (4.18) and (4.19), that is, the set of equilibrium points of (1.1) and (1.2),

$$\begin{aligned} \mathcal{E}_\epsilon &= \{u_\epsilon \in X^\alpha : A_{-\beta}u_\epsilon - (F_\epsilon)_\Omega(u_\epsilon) = 0\}, \quad 0 < \epsilon \leq \epsilon_0 \\ \mathcal{E}_0 &= \{u_0 \in X^\alpha : A_{-\beta}u_0 - (F_0)_\gamma(u_0) = 0\}. \end{aligned}$$

The following lemmas will be important to prove the continuity of the family of equilibria $\{\mathcal{E}_\epsilon\}_{\epsilon \in [0, \epsilon_0]}$ at $\epsilon = 0$.

Lemma 4.2. *Suppose that (H4) holds and that α , β and p satisfy (2.4).*

- (1) *If $u \in X^\alpha$ satisfies $\|u\|_{C(\bar{\Omega})} \leq R$, then there exists $K > 0$ independent of $\epsilon > 0$ such that*

$$\sup_u \{ \|(F_0)_\gamma(u)\|_{X^{-\beta}}, \|(F_\epsilon)_\Omega(u)\|_{X^{-\beta}} \} \leq K.$$

- (2) *The maps $(F_0)_\gamma, (F_\epsilon)_\Omega : X^\alpha \rightarrow X^{-\beta}$, $0 < \epsilon \leq \epsilon_0$, are locally Lipschitz, uniformly in ϵ .*
 (3) *For each $u \in X^\alpha$,*

$$\|(F_\epsilon)_\Omega(u) - (F_0)_\gamma(u)\|_{X^{-\beta}} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

Furthermore, this limit is uniform for $u \in X^\alpha$ such that $\|u\|_{X^\alpha} \leq R$, for some $R > 0$.

- (4) *If $u_\epsilon \rightarrow u$ in X^α , as $\epsilon \rightarrow 0$, then*

$$\|(F_\epsilon)_\Omega(u_\epsilon) - (F_0)_\gamma(u)\|_{X^{-\beta}} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

Proof. 1. Let $u \in X^\alpha$ such that $\|u\|_{C(\bar{\Omega})} \leq R$. For each $\phi \in H_{p'}^{2\beta}(\Omega)$ and $0 < \epsilon \leq \epsilon_0$, from (3.11) we have

$$| \langle (F_\epsilon)_\Omega(u), \phi \rangle | \leq C \left(\frac{1}{\epsilon} \int_{\omega_\epsilon} |f(u(x), u(x))|^p dx \right)^{\frac{1}{p}} \|\phi\|_{H_{p'}^{2\beta}(\Omega)}.$$

Using that f is continuous, we have that there exists $\bar{K} = \bar{K}(R) > 0$ such that

$$\|(F_\epsilon)_\Omega(u)\|_{X^{-\beta}} \leq C\bar{K} \left(\frac{|\omega_\epsilon|}{\epsilon} \right)^{\frac{1}{p}} \leq K, \quad \forall 0 < \epsilon \leq \epsilon_0,$$

where $K = K(R) > 0$ does not depend of ϵ , we note that $|\omega_\epsilon| \leq k|\partial\Omega|\epsilon$, for some $k > 0$ independent of ϵ . Similarly, there exists $K = K(R) > 0$ such that $\|(F_0)_\gamma(u)\|_{X^{-\beta}} \leq K$.

2. Let $u, v \in X^\alpha$ such that $\|u\|_{X^\alpha}, \|v\|_{X^\alpha} \leq \rho$, for some $\rho > 0$. For each $0 < \epsilon \leq \epsilon_0$ and $\phi \in H_{p'}^{2\beta}(\Omega)$, from (3.11) we have

$$\begin{aligned} & | \langle (F_\epsilon)_\Omega(u) - (F_\epsilon)_\Omega(v), \phi \rangle | \\ & \leq C \left(\frac{1}{\epsilon} \int_{\omega_\epsilon} |f(u(x), u(x)) - f(v(x), v(x))|^p dx \right)^{\frac{1}{p}} \|\phi\|_{H_{p'}^{2\beta}(\Omega)}. \end{aligned}$$

Using (H4) and $X^\alpha \hookrightarrow C(\bar{\Omega})$, we have that there exists $\bar{K} = \bar{K}(\rho) > 0$ such that

$$\begin{aligned} & \| (F_\epsilon)_\Omega(u) - (F_\epsilon)_\Omega(v) \|_{X^{-\beta}} \\ & \leq 2C \left(\frac{1}{\epsilon} \int_{\omega_\epsilon} |df(\theta(x)(u(x), u(x)) + (1 - \theta(x))(v(x), v(x)))|^p \right. \\ & \quad \left. |u(x) - v(x)|^p dx \right)^{\frac{1}{p}} \\ & \leq 2C\bar{K} \left(\frac{|\omega_\epsilon|}{\epsilon} \right)^{\frac{1}{p}} \|u - v\|_{X^\alpha}, \end{aligned}$$

for some $0 \leq \theta(x) \leq 1$, $x \in \bar{\Omega}$. Hence, there exists $L = L(\rho) > 0$ independent of ϵ such that

$$\| (F_\epsilon)_\Omega(u) - (F_\epsilon)_\Omega(v) \|_{X^{-\beta}} \leq L \|u - v\|_{X^\alpha}.$$

Therefore, for each $0 < \epsilon \leq \epsilon_0$, $(F_\epsilon)_\Omega$ is locally Lipschitz, uniformly in ϵ . Similarly, $(F_0)_\gamma$ is locally Lipschitz.

3. Initially, we take α_0 and β_0 satisfying (2.4). For each $\phi \in H_{p'}^{2\beta_0}(\Omega)$ and $u \in X^{\alpha_0}$, we have

$$\begin{aligned} & | \langle (F_\epsilon)_\Omega(u), \phi \rangle - \langle (F_0)_\gamma(u), \phi \rangle | = \\ & \left| \frac{1}{\epsilon} \int_{\omega_\epsilon} f(u(x), u(x))\phi(x)dx - \int_{\partial\Omega} \gamma(f(u(x), u(x))) \gamma(\phi(x)) dx \right|. \end{aligned}$$

From [6, Lemma 2.1] we get

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{\omega_\epsilon} f(u(x), u(x))\phi(x)dx = \int_{\partial\Omega} \gamma(f(u(x), u(x))) \gamma(\phi(x)) dx.$$

Thus, for each $\phi \in H_{p'}^{2\beta_0}(\Omega)$ and $u \in X^{\alpha_0}$,

$$\langle (F_\epsilon)_\Omega(u), \phi \rangle \rightarrow \langle (F_0)_\gamma(u), \phi \rangle, \quad \text{as } \epsilon \rightarrow 0. \quad (4.20)$$

Moreover, fixed $u \in X^{\alpha_0}$, the set $\{ (F_\epsilon)_\Omega(u) \in X^{-\beta_0} : \epsilon \in (0, \epsilon_0] \}$ is equicontinuous. Thus, the limit (4.20) is uniform for ϕ in compact sets of

$H_{p'}^{2\beta_0}(\Omega)$. Hence, choosing β_0 such that $\beta > \beta_0$, with $2\beta > \frac{1}{p'}$, we have that the embedding $H_{p'}^{2\beta}(\Omega) \hookrightarrow H_{p'}^{2\beta_0}(\Omega)$ is compact, and then, in particular,

$$\|(F_\epsilon)_\Omega(u) - (F_0)_\gamma(u)\|_{X^{-\beta}} = \sup_{\|\phi\|_{H_{p'}^{2\beta}(\Omega)}=1} |\langle (F_\epsilon)_\Omega(u) - (F_0)_\gamma(u), \phi \rangle| \rightarrow 0, \quad (4.21)$$

as $\epsilon \rightarrow 0$.

Now, the set $\{(F_\epsilon)_\Omega : X^{\alpha_0} \rightarrow X^{-\beta} : \epsilon \in (0, \epsilon_0]\}$ is equicontinuous. Thus, the limit (4.21) is uniform for u in compact sets of X^{α_0} . Hence, choosing α_0 such that $\alpha > \alpha_0$, with $2\alpha > \frac{1}{p}$, we have that the embedding $X^\alpha \hookrightarrow X^{\alpha_0}$ is compact, and then, in particular, the limit (4.21) is uniform for $u \in X^\alpha$ such that $\|u\|_{X^\alpha} \leq R$.

4. This item follows from 2. and 3. adding and subtracting $(F_\epsilon)_\Omega(u)$. \square

Using the hypothesis (H4) we can show that $(F_0)_\gamma, (F_\epsilon)_\Omega : X^\alpha \rightarrow X^{-\beta}$, $0 < \epsilon \leq \epsilon_0$, are Fréchet differentiable, uniformly in ϵ , and your Fréchet differential are given, respectively, by

$$\begin{aligned} (F_0)_\gamma' : X^\alpha &\rightarrow \mathcal{L}(X^\alpha, X^{-\beta}) \\ u^* &\mapsto (F_0)_\gamma'(u^*) : X^\alpha \rightarrow X^{-\beta} \\ w &\mapsto (F_0)_\gamma'(u^*)w \end{aligned}$$

$$\langle (F_0)_\gamma'(u^*)w, \phi \rangle := \int_{\partial\Omega} \gamma(df(u^*(x), u^*(x))(w(x), w(x))) \gamma(\phi(x)) dx,$$

$\forall \phi \in H_{p'}^{2\beta}(\Omega)$, where γ denotes the trace operator and $\mathcal{L}(X^\alpha, X^{-\beta})$ denotes the space of the continuous linear operators from X^α in $X^{-\beta}$, and

$$\begin{aligned} (F_\epsilon)_\Omega' : X^\alpha &\rightarrow \mathcal{L}(X^\alpha, X^{-\beta}) \\ u^* &\mapsto (F_\epsilon)_\Omega'(u^*) : X^\alpha \rightarrow X^{-\beta} \\ w &\mapsto (F_\epsilon)_\Omega'(u^*)w \end{aligned}$$

$$\langle (F_\epsilon)_\Omega'(u^*)w, \phi \rangle := \frac{1}{\epsilon} \int_{\omega_\epsilon} df(u^*(x), u^*(x))(w(x), w(x))\phi(x) dx,$$

$\forall \phi \in H_{p'}^{2\beta}(\Omega)$.

Similarly to Lemma 4.2, we can prove the following lemma:

Lemma 4.3. *Suppose that (H4) holds and that α, β and p satisfy (2.4).*

- (1) If $u^* \in X^\alpha$ satisfies $\|u^*\|_{C(\bar{\Omega})} \leq R$, then there exists $K > 0$ independent of $\epsilon > 0$ such that

$$\sup_u \left\{ \|(F_0)'_\gamma(u^*)\|_{\mathcal{L}(X^\alpha, X^{-\beta})}, \|(F_\epsilon)'_\Omega(u^*)\|_{\mathcal{L}(X^\alpha, X^{-\beta})} \right\} \leq K.$$

- (2) The maps $(F_0)'_\gamma, (F_\epsilon)'_\Omega : X^\alpha \rightarrow \mathcal{L}(X^\alpha, X^{-\beta}), 0 < \epsilon \leq \epsilon_0$, are locally Lipschitz, uniformly in ϵ .

- (3) For each $u^* \in X^\alpha$,

$$\|(F_\epsilon)'_\Omega(u^*) - (F_0)'_\gamma(u^*)\|_{\mathcal{L}(X^\alpha, X^{-\beta})} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

- (4) If $u_\epsilon \rightarrow u^*$ in X^α , as $\epsilon \rightarrow 0$, then

$$\|(F_\epsilon)'_\Omega(u_\epsilon) - (F_0)'_\gamma(u^*)\|_{\mathcal{L}(X^\alpha, X^{-\beta})} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

- (5) If $u_\epsilon \rightarrow u^*$ in X^α , as $\epsilon \rightarrow 0$, and $w_\epsilon \rightarrow w$ in X^α , as $\epsilon \rightarrow 0$, then

$$\|(F_\epsilon)'_\Omega(u_\epsilon)w_\epsilon - (F_0)'_\gamma(u^*)w\|_{X^{-\beta}} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

4.2. Lower semicontinuity of the set of equilibria. In order to obtain continuity of the family of equilibria $\{\mathcal{E}_\epsilon\}_{\epsilon \in [0, \epsilon_0]}$ of (1.1) and (1.2) at $\epsilon = 0$, we need to prove the lower semicontinuity of this family at $\epsilon = 0$. For this, we will use the fact that $\mathcal{E}_\epsilon \subset \mathcal{A}_\epsilon$, for all $\epsilon \in [0, \epsilon_0]$, and that the attractor \mathcal{A}_ϵ is bounded in C_α , uniformly in ϵ (Lemma 3.9), consequently, the set of equilibria \mathcal{E}_ϵ is bounded in X^α , uniformly in ϵ , since the elements of \mathcal{E}_ϵ are the solutions of (1.1) and (1.2) constants in the time.

Initially, we will show that each set of equilibria is not empty and it is compact for $\epsilon_0 > 0$ sufficiently small. First, we note the following:

Remark 4.4. The linear operator $A_{-\beta} : X^{1-\beta} \subset X^{-\beta} \rightarrow X^{-\beta}$ is closed, densely defined, sectorial and positive, with compact resolvent set $\rho(A_{-\beta})$ in $X^{-\beta}$, in particular, $A_{-\beta}^{-1} : X^{-\beta} \rightarrow X^{1-\beta}$ is continuous. Since α, β and p satisfy (2.4), then the embedding of $X^{1-\beta}$ in X^α is compact, hence we get that $A_{-\beta}^{-1} : X^{-\beta} \rightarrow X^\alpha$ is compact.

Lemma 4.5. Suppose that (H2), (H3) and (H4) hold and that α, β and p satisfy (2.4). Then, for each $\epsilon \in [0, \epsilon_0]$, the set \mathcal{E}_ϵ of the solutions of (4.18) and (4.19) is not empty. Moreover, \mathcal{E}_ϵ is compact in X^α , for $\epsilon \in [0, \epsilon_0]$, with $\epsilon_0 > 0$ sufficiently small.

Proof. As we have proved in the Section 3, for each $\epsilon \in [0, \epsilon_0]$, the semigroup $\{U^\epsilon(t) : t \geq 0\}$, associated to solutions of the problems (1.1) and (1.2), is asymptotically smooth, point dissipative and orbits of bounded set of C_α are bounded in C_α . Hence, from [9, Theorem 5.1] we have that there exists an equilibrium point of $U^\epsilon(t)$.

Now, let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence in \mathcal{E}_0 , then $u_n = A_{-\beta}^{-1}(F_0)_\gamma(u_n)$, for all $n \in \mathbb{N}$. Since $\mathcal{E}_0 \subset \mathcal{A}_0$ and, by Lemma 3.9, there exists $R > 0$ such that $\mathcal{A}_0 \subset \bar{B}_R(0)$, where $\bar{B}_R(0)$ is the closed ball in C_α with center in the origin and ray R , then $\|u_n\|_{X^\alpha} \leq R$, for all $n \in \mathbb{N}$. Thus, $\{u_n\}_{n \in \mathbb{N}}$ is a bounded sequence in X^α and $X^\alpha \hookrightarrow C(\bar{\Omega})$. From item 1 of Lemma 4.2, $\{(F_0)_\gamma(u_n)\}_{n \in \mathbb{N}}$ is a bounded family in $X^{-\beta}$. From Remark 4.4, $A_{-\beta}^{-1} : X^{-\beta} \rightarrow X^\alpha$ is compact, hence we have that $\{A_{-\beta}^{-1}(F_0)_\gamma(u_n)\}_{n \in \mathbb{N}}$ has a convergent subsequence, that we will denote by $\{A_{-\beta}^{-1}(F_0)_\gamma(u_{n_k})\}_{k \in \mathbb{N}}$, with limit $u \in X^\alpha$, that is,

$$A_{-\beta}^{-1}(F_0)_\gamma(u_{n_k}) \rightarrow u \quad \text{in } X^\alpha, \quad \text{as } k \rightarrow \infty.$$

Hence, $u_{n_k} \rightarrow u$ in X^α , as $k \rightarrow \infty$. By the continuity of the operator $A_{-\beta}^{-1}(F_0)_\gamma : X^\alpha \rightarrow X^\alpha$, we get

$$A_{-\beta}^{-1}(F_0)_\gamma(u_{n_k}) \rightarrow A_{-\beta}^{-1}(F_0)_\gamma(u) \quad \text{in } X^\alpha, \quad \text{as } k \rightarrow \infty.$$

By the uniqueness of the limit, $u = A_{-\beta}^{-1}(F_0)_\gamma(u)$. Thus, $A_{-\beta}u - (F_0)_\gamma(u) = 0$ and $u \in \mathcal{E}_0$. Therefore, \mathcal{E}_0 is a compact set in X^α . Similarly, since $\mathcal{E}_\epsilon \subset \mathcal{A}_\epsilon$, $\epsilon \in (0, \epsilon_0]$ and, by Lemma 3.9, there exists $R > 0$ independent of ϵ such that $\mathcal{A}_\epsilon \subset \bar{B}_R(0)$, for all $\epsilon \in (0, \epsilon_0]$, with $\epsilon_0 > 0$ sufficiently small, then \mathcal{E}_ϵ is a compact set in X^α . \square

The proof of lower semicontinuity of the family of equilibria $\{\mathcal{E}_\epsilon\}_{\epsilon \in [0, \epsilon_0]}$ at $\epsilon = 0$, requires additional assumptions. We need to assume that the equilibrium points of (1.2) are stable under perturbation. This stability under perturbation can be given excluding the zero of spectrum or by the hyperbolicity.

Definition 4.6. We say that the solutions u_0^* of (4.19) and u_ϵ^* , $0 < \epsilon \leq \epsilon_0$, of (4.18) are hyperbolic if the spectrums $\sigma(A_{-\beta} - (F_0)_\gamma'(u_0^*))$ and $\sigma(A_{-\beta} - (F_\epsilon)_\Omega'(u_\epsilon^*))$ are disjoint from the imaginary axis, that is, $\sigma(A_{-\beta} - (F_0)_\gamma'(u_0^*)) \cap i\mathbb{R} = \emptyset$ and $\sigma(A_{-\beta} - (F_\epsilon)_\Omega'(u_\epsilon^*)) \cap i\mathbb{R} = \emptyset$, $0 < \epsilon \leq \epsilon_0$.

Theorem 4.7. Suppose that (H4) holds and that α , β and p satisfy (2.4). If u_0^* is a solution of (4.19), that is, an equilibrium point of (1.2), which satisfies $0 \notin \sigma(A_{-\beta} - (F_0)_\gamma'(u_0^*))$, then u_0^* is isolated.

Proof. Since $0 \notin \sigma(A_{-\beta} - (F_0)_\gamma'(u_0^*))$ then $0 \in \rho(A_{-\beta} - (F_0)_\gamma'(u_0^*))$. Thus, there exists $C > 0$ such that

$$\|(A_{-\beta} - (F_0)_\gamma'(u_0^*))^{-1}\|_{\mathcal{L}(X^{-\beta}, X^\alpha)} \leq C.$$

Now, we note that u is a solution of (4.19) if and only if

$$\begin{aligned} 0 &= A_{-\beta}u - (F_0)'_{\gamma}(u_0^*)u + (F_0)'_{\gamma}(u_0^*)u - (F_0)_{\gamma}(u) \\ &\Leftrightarrow u = (A_{-\beta} - (F_0)'_{\gamma}(u_0^*))^{-1} ((F_0)_{\gamma}(u) - (F_0)'_{\gamma}(u_0^*)u). \end{aligned}$$

So, u is a solution of (4.19) if and only if u is a fixed point of the map

$$\begin{aligned} \Phi : X^{\alpha} &\longrightarrow X^{\alpha} \\ u &\longmapsto \Phi(u) = (A_{-\beta} - (F_0)'_{\gamma}(u_0^*))^{-1} ((F_0)_{\gamma}(u) - (F_0)'_{\gamma}(u_0^*)u). \end{aligned}$$

We will show that there exists $r > 0$ such that $\Phi : \bar{B}_r(u_0^*) \longrightarrow \bar{B}_r(u_0^*)$ is a contraction, where $\bar{B}_r(u_0^*)$ is a closed ball in X^{α} with center in u_0^* and ray r .

In fact, since $(F_0)_{\gamma}$ is Fréchet differentiable, then there exists $\delta > 0$ such that

$$C \|(F_0)_{\gamma}(u) - (F_0)_{\gamma}(v) - (F_0)'_{\gamma}(u_0^*)(u - v)\|_{X^{-\beta}} \leq \frac{1}{2} \|u - v\|_{X^{\alpha}},$$

for $\|u - v\|_{X^{\alpha}} \leq \delta$.

Taking $r = \frac{\delta}{2}$ and $u, v \in \bar{B}_r(u_0^*)$, we have

$$\begin{aligned} &\|\Phi(u) - \Phi(v)\|_{X^{\alpha}} \\ &= \|(A_{-\beta} - (F_0)'_{\gamma}(u_0^*))^{-1} [(F_0)_{\gamma}(u) - (F_0)_{\gamma}(v) - (F_0)'_{\gamma}(u_0^*)(u - v)]\|_{X^{\alpha}} \\ &\leq C \|(F_0)_{\gamma}(u) - (F_0)_{\gamma}(v) - (F_0)'_{\gamma}(u_0^*)(u - v)\|_{X^{-\beta}} \leq \frac{1}{2} \|u - v\|_{X^{\alpha}}. \end{aligned}$$

Thus, Φ is a contraction on the $\bar{B}_r(u_0^*)$. Moreover, if $u \in \bar{B}_r(u_0^*)$ then

$$\|\Phi(u) - u_0^*\|_{X^{\alpha}} = \|\Phi(u) - \Phi(u_0^*)\|_{X^{\alpha}} \leq \frac{1}{2} \|u - u_0^*\|_{X^{\alpha}} \leq \frac{1}{2} r < r.$$

Hence, $\Phi(\bar{B}_r(u_0^*)) \subset \bar{B}_r(u_0^*)$.

Therefore, from Contraction Theorem, Φ has an unique fixed point in $\bar{B}_r(u_0^*)$. Since u_0^* is a fixed point of Φ , then u_0^* is the unique fixed point of Φ in $\bar{B}_r(u_0^*)$. Thus, u_0^* is isolated. \square

Corollary 4.8. *Suppose that (H4) holds and that α, β and p satisfy (2.4). If u_0^* is a hyperbolic solution of (4.19), then u_0^* is an isolated equilibrium point.*

Proposition 4.9. *Suppose that (H2), (H3) and (H4) hold and that α, β and p satisfy (2.4). If all points in \mathcal{E}_0 are isolated, then there is only a finite*

number of them. Moreover, if $0 \notin \sigma(A_{-\beta} - (F_0)'_{\gamma}(u_0^*))$ for each $u_0^* \in \mathcal{E}_0$, then \mathcal{E}_0 is a finite set.

Proof. We suppose that the number of elements in \mathcal{E}_0 is infinite, hence there exists a sequence $\{u_n\}_{n \in \mathbb{N}}$ in \mathcal{E}_0 . From Lemma 4.5, \mathcal{E}_0 is compact, thus there exist a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ of $\{u_n\}_{n \in \mathbb{N}}$ and $u^* \in \mathcal{E}_0$ such that $u_{n_k} \rightarrow u^*$ in X^α , as $k \rightarrow \infty$. Thus, for all $\delta > 0$, there exists $k_0 \in \mathbb{N}$ such that $u_{n_k} \in B_\delta(u^*)$, for all $k > k_0$, which is a contradiction with the fact that each fixed point in \mathcal{E}_0 is isolated and $u^* \in \mathcal{E}_0$.

If $0 \notin \sigma(A_{-\beta} - (F_0)'_{\gamma}(u_0^*))$ for each $u_0^* \in \mathcal{E}_0$, then, by Theorem 4.7, u_0^* is isolated. Thus, \mathcal{E}_0 is a finite set. \square

To prove the lower semicontinuity of the family of equilibria $\{\mathcal{E}_\epsilon\}_{\epsilon \in [0, \epsilon_0]}$, we will need of the following lemmas:

Lemma 4.10. *Suppose that (H4) holds, that α , β and p satisfy (2.4) and let $u^* \in X^\alpha$ such that $\|u^*\|_{C(\bar{\Omega})} \leq R$. Then, the operators $A_{-\beta}^{-1}(F_0)'_{\gamma}(u^*)$, $A_{-\beta}^{-1}(F_\epsilon)'_{\Omega}(u^*) : X^\alpha \rightarrow X^\alpha$, $\epsilon \in (0, \epsilon_0]$, are compact. For any bounded family $\{w_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$ in X^α , the family $\{A_{-\beta}^{-1}(F_\epsilon)'_{\Omega}(u^*)w_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$ is relatively compact in X^α . Moreover, if $w_\epsilon \rightarrow w$ in X^α , as $\epsilon \rightarrow 0$, then*

$$A_{-\beta}^{-1}(F_\epsilon)'_{\Omega}(u^*)w_\epsilon \rightarrow A_{-\beta}^{-1}(F_0)'_{\gamma}(u^*)w \quad \text{in } X^\alpha, \quad \text{as } \epsilon \rightarrow 0.$$

Proof. The compactness of the linear operators $A_{-\beta}^{-1}(F_0)'_{\gamma}(u^*)$, $A_{-\beta}^{-1}(F_\epsilon)'_{\Omega}(u^*) : X^\alpha \rightarrow X^\alpha$, $\epsilon \in (0, \epsilon_0]$, follows from item 1 of Lemma 4.3 and of compactness of the linear operator $A_{-\beta}^{-1} : X^{-\beta} \rightarrow X^\alpha$.

Let $\{w_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$ be a bounded family in X^α . Since

$$\|(F_\epsilon)'_{\Omega}(u^*)w_\epsilon\|_{X^{-\beta}} \leq \|(F_\epsilon)'_{\Omega}(u^*)\|_{\mathcal{L}(X^\alpha, X^{-\beta})} \|w_\epsilon\|_{X^\alpha}, \quad \forall \epsilon \in (0, \epsilon_0],$$

and from item 1 of Lemma 4.3, $\{(F_\epsilon)'_{\Omega}(u^*)\}_{\epsilon \in (0, \epsilon_0]}$ is a bounded family in $\mathcal{L}(X^\alpha, X^{-\beta})$, uniformly in ϵ , then $\{(F_\epsilon)'_{\Omega}(u^*)w_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$ is a bounded family in $X^{-\beta}$. By compactness of $A_{-\beta}^{-1} : X^{-\beta} \rightarrow X^\alpha$, we have that $\{A_{-\beta}^{-1}(F_\epsilon)'_{\Omega}(u^*)w_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$ has a convergent subsequence in X^α . Therefore, the family $\{A_{-\beta}^{-1}(F_\epsilon)'_{\Omega}(u^*)w_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$ is relatively compact in X^α .

Now, let us take $w_\epsilon \rightarrow w$ in X^α , as $\epsilon \rightarrow 0$. Thus, from item 5 of Lemma 4.3,

$$(F_\epsilon)'_{\Omega}(u^*)w_\epsilon \rightarrow (F_0)'_{\gamma}(u^*)w \quad \text{in } X^{-\beta}, \quad \text{as } \epsilon \rightarrow 0.$$

By the continuity of the operator $A_{-\beta}^{-1} : X^{-\beta} \longrightarrow X^{\alpha}$, we get

$$A_{-\beta}^{-1}(F_{\epsilon})'_{\Omega}(u^*)w_{\epsilon} \rightarrow A_{-\beta}^{-1}(F_0)'_{\gamma}(u^*)w \quad \text{in } X^{\alpha}, \quad \text{as } \epsilon \rightarrow 0.$$

□

Lemma 4.11. *Suppose that (H4) holds, that α, β and p satisfy (2.4) and let $u^* \in X^{\alpha}$ such that $\|u^*\|_{C(\bar{\Omega})} \leq R$ and $0 \notin \sigma(A_{-\beta} - (F_0)'_{\gamma}(u^*))$. Then, there exist $\epsilon_0 > 0$ and $C > 0$ independent of ϵ such that $0 \notin \sigma(A_{-\beta} - (F_{\epsilon})'_{\Omega}(u^*))$ and*

$$\|(A_{-\beta} - (F_{\epsilon})'_{\Omega}(u^*))^{-1}\|_{\mathcal{L}(X^{-\beta}, X^{\alpha})} \leq C, \quad \forall \epsilon \in (0, \epsilon_0]. \quad (4.22)$$

Furthermore, the operators $(A_{-\beta} - (F_0)'_{\gamma}(u^*))^{-1}, (A_{-\beta} - (F_{\epsilon})'_{\Omega}(u^*))^{-1} : X^{-\beta} \longrightarrow X^{\alpha}, \epsilon \in (0, \epsilon_0]$, are compact. For any bounded family $\{w_{\epsilon}\}_{\epsilon \in (0, \epsilon_0]}$ in $X^{-\beta}$, the family $\{(A_{-\beta} - (F_{\epsilon})'_{\Omega}(u^*))^{-1}w_{\epsilon}\}_{\epsilon \in (0, \epsilon_0]}$ is relatively compact in X^{α} . Moreover, if $w_{\epsilon} \rightarrow w$ in $X^{-\beta}$, as $\epsilon \rightarrow 0$, then

$$(A_{-\beta} - (F_{\epsilon})'_{\Omega}(u^*))^{-1}w_{\epsilon} \rightarrow (A_{-\beta} - (F_0)'_{\gamma}(u^*))^{-1}w \quad \text{in } X^{\alpha}, \quad \text{as } \epsilon \rightarrow 0.$$

Proof. Initially, we note that

$$\begin{aligned} (A_{-\beta} - (F_{\epsilon})'_{\Omega}(u^*))^{-1} &= \left[A_{-\beta} (I - A_{-\beta}^{-1} (F_{\epsilon})'_{\Omega}(u^*)) \right]^{-1} \\ &= (I - A_{-\beta}^{-1} (F_{\epsilon})'_{\Omega}(u^*))^{-1} A_{-\beta}^{-1}, \quad \epsilon \in (0, \epsilon_0]. \end{aligned}$$

Then, prove that $0 \notin \sigma(A_{-\beta} - (F_{\epsilon})'_{\Omega}(u^*))$ it is equivalent to prove that $1 \in \rho(A_{-\beta}^{-1} (F_{\epsilon})'_{\Omega}(u^*))$. Moreover, to prove that there exist $\epsilon_0 > 0$ and $C > 0$ independent of ϵ such that (4.22) holds, it is enough to prove that there exist $\epsilon_0 > 0$ and $M > 0$ independent of ϵ such that

$$\|(I - A_{-\beta}^{-1} (F_{\epsilon})'_{\Omega}(u^*))^{-1}\|_{\mathcal{L}(X^{\alpha})} \leq M, \quad \forall \epsilon \in (0, \epsilon_0]. \quad (4.23)$$

Then, we will show (4.23). Initially, from hypothesis $0 \notin \sigma(A_{-\beta} - (F_0)'_{\gamma}(u^*))$, thus $1 \in \rho(A_{-\beta}^{-1} (F_0)'_{\gamma}(u^*))$. Hence, there exists the inverse $(I - A_{-\beta}^{-1} (F_0)'_{\gamma}(u^*))^{-1} : X^{\alpha} \longrightarrow X^{\alpha}$, in particular, the kernel $\mathcal{N}(I - A_{-\beta}^{-1} (F_0)'_{\gamma}(u^*)) = \{0\}$.

Now, let $B_0 = A_{-\beta}^{-1} (F_0)'_{\gamma}(u^*)$ and $B_{\epsilon} = A_{-\beta}^{-1} (F_{\epsilon})'_{\Omega}(u^*), \epsilon \in (0, \epsilon_0]$. From Lemma 4.10 we have that, for each $\epsilon \in [0, \epsilon_0]$ fixed, the operator $B_{\epsilon} : X^{\alpha} \longrightarrow X^{\alpha}$ is compact. Using the compactness of B_{ϵ} , we can show that estimate (4.23) is equivalent to say

$$\|(I - B_{\epsilon})u_{\epsilon}\|_{X^{\alpha}} \geq \frac{1}{M}, \quad (4.24)$$

$\forall \epsilon \in (0, \epsilon_0]$ and $\forall u_\epsilon \in X^\alpha$ with $\|u_\epsilon\|_{X^\alpha} = 1$.

In fact, suppose that (4.23) holds, then there exists the inverse $(I - B_\epsilon)^{-1} : X^\alpha \rightarrow X^\alpha$ and it is continuous. Moreover,

$$\|(I - B_\epsilon)^{-1}v_\epsilon\|_{X^\alpha} \leq M \|v_\epsilon\|_{X^\alpha}, \quad \forall \epsilon \in (0, \epsilon_0] \quad \text{and} \quad \forall v_\epsilon \in X^\alpha.$$

Let $u_\epsilon \in X^\alpha$ such that $\|u_\epsilon\|_{X^\alpha} = 1$ and taking $v_\epsilon = (I - B_\epsilon)u_\epsilon$, we have

$$\|(I - B_\epsilon)^{-1}(I - B_\epsilon)u_\epsilon\|_{X^\alpha} \leq M \|(I - B_\epsilon)u_\epsilon\|_{X^\alpha} \Rightarrow \|(I - B_\epsilon)u_\epsilon\|_{X^\alpha} \geq \frac{1}{M}.$$

Therefore, (4.24) holds. Reversely, suppose that (4.24) holds. We want to prove that there exists the inverse $(I - B_\epsilon)^{-1} : X^\alpha \rightarrow X^\alpha$, it is continuous and satisfies (4.23). First, we will prove the following estimative

$$\|(I - B_\epsilon)u_\epsilon\|_{X^\alpha} \geq \frac{1}{M} \|u_\epsilon\|_{X^\alpha}, \quad \forall \epsilon \in (0, \epsilon_0] \quad \text{and} \quad \forall u_\epsilon \in X^\alpha. \quad (4.25)$$

We note that (4.25) is immediate for $u_\epsilon = 0$. Now, let $u_\epsilon \in X^\alpha$ such that $u_\epsilon \neq 0$, then (4.25) also holds. In fact, taking $v_\epsilon = \frac{u_\epsilon}{\|u_\epsilon\|_{X^\alpha}}$ we have $\|v_\epsilon\|_{X^\alpha} = 1$ and using (4.24), we get

$$\begin{aligned} \|(I - B_\epsilon)v_\epsilon\|_{X^\alpha} \geq \frac{1}{M} &\Rightarrow \left\| (I - B_\epsilon) \frac{u_\epsilon}{\|u_\epsilon\|_{X^\alpha}} \right\|_{X^\alpha} \geq \frac{1}{M} \\ &\Rightarrow \|(I - B_\epsilon)u_\epsilon\|_{X^\alpha} \geq \frac{1}{M} \|u_\epsilon\|_{X^\alpha}. \end{aligned}$$

Now, let $u_\epsilon \in X^\alpha$ such that $(I - B_\epsilon)u_\epsilon = 0$. From (4.25) follows $u_\epsilon = 0$. Thus, for each $\epsilon \in (0, \epsilon_0]$, $\mathcal{N}(I - B_\epsilon) = \{0\}$ and the operator $I - B_\epsilon$ is injective. So, there exists the inverse $(I - B_\epsilon)^{-1} : \mathcal{R}(I - B_\epsilon) \rightarrow X^\alpha$, where $\mathcal{R}(I - B_\epsilon)$ denotes the image of the operator $I - B_\epsilon$.

Since for each $\epsilon \in (0, \epsilon_0]$, B_ϵ is compact and $\mathcal{N}(I - B_\epsilon) = \{0\}$, using the Fredholm Alternative Theorem, we have $\mathcal{R}(I - B_\epsilon) = X^\alpha$ and $I - B_\epsilon$ is bijective. Thus, there exists the inverse $(I - B_\epsilon)^{-1} : X^\alpha \rightarrow X^\alpha$.

Taking $v_\epsilon \in X^\alpha$ we have that there exists $u_\epsilon \in X^\alpha$ such that $(I - B_\epsilon)u_\epsilon = v_\epsilon$ and $u_\epsilon = (I - B_\epsilon)^{-1}v_\epsilon$. From (4.25) we have

$$\begin{aligned} \|(I - B_\epsilon)^{-1}v_\epsilon\|_{X^\alpha} &= \|u_\epsilon\|_{X^\alpha} \\ &\leq M \|(I - B_\epsilon)u_\epsilon\|_{X^\alpha} = M \|v_\epsilon\|_{X^\alpha}, \quad \forall \epsilon \in (0, \epsilon_0] \quad \text{and} \quad \forall v_\epsilon \in X^\alpha. \\ &\Rightarrow \|(I - B_\epsilon)^{-1}\|_{\mathcal{L}(X^\alpha)} \leq M, \quad \forall \epsilon \in (0, \epsilon_0]. \end{aligned}$$

Therefore, (4.23) holds.

Since (4.23) and (4.24) are equivalents, then it is enough to show (4.24). Suppose that (4.24) is not true, that is, there exist a sequence $\{u_n\}_{n \in \mathbb{N}}$ in X^α , with $\|u_n\|_{X^\alpha} = 1$ and $\epsilon_n \rightarrow 0$, as $n \rightarrow \infty$, such that

$$\|(I - B_{\epsilon_n})u_n\|_{X^\alpha} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

From Lemma 4.10 we get that $\{B_{\epsilon_n}u_n\}_{n \in \mathbb{N}}$ is relatively compact. Thus, $\{B_{\epsilon_n}u_n\}_{n \in \mathbb{N}}$ has a convergent subsequence, which we again denote by $\{B_{\epsilon_n}u_n\}_{n \in \mathbb{N}}$, with limit $u \in X^\alpha$, that is,

$$B_{\epsilon_n}u_n \rightarrow u \quad \text{in } X^\alpha, \quad \text{as } n \rightarrow \infty.$$

Since $u_n - B_{\epsilon_n}u_n \rightarrow 0$ in X^α , as $n \rightarrow \infty$, then $u_n \rightarrow u$ in X^α , as $n \rightarrow \infty$. Hence, $\|u\|_{X^\alpha} = 1$. Moreover, using the Lemma 4.10 we have that $B_{\epsilon_n}u_n \rightarrow B_0u$ in X^α , as $n \rightarrow \infty$. Thus,

$$u_n - B_{\epsilon_n}u_n \rightarrow u - B_0u \quad \text{in } X^\alpha, \quad \text{as } n \rightarrow \infty.$$

By the uniqueness of the limit, $u - B_0u = 0$. This implies that $(I - B_0)u = 0$, with $u \neq 0$, contradicting our hypothesis. Therefore, (4.24) holds.

With this, we conclude that there exist $\epsilon_0 > 0$ and $C > 0$ independent of ϵ such that (4.22) holds.

Now, the operators $(A_{-\beta} - (F_0)'_\gamma(u^*))^{-1}$, $(A_{-\beta} - (F_\epsilon)'_\Omega(u^*))^{-1}$, $\epsilon \in (0, \epsilon_0]$, are compact and the prove of this compactness follows similarly to account below.

Let $\{w_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$ be a bounded family in $X^{-\beta}$. For each $\epsilon \in (0, \epsilon_0]$, let

$$v_\epsilon = (A_{-\beta} - (F_\epsilon)'_\Omega(u^*))^{-1} w_\epsilon.$$

From (4.22) we have

$$\begin{aligned} \|v_\epsilon\|_{X^\alpha} &\leq \|(A_{-\beta} - (F_\epsilon)'_\Omega(u^*))^{-1} w_\epsilon\|_{X^\alpha} \\ &\leq \|(A_{-\beta} - (F_\epsilon)'_\Omega(u^*))^{-1}\|_{\mathcal{L}(X^{-\beta}, X^\alpha)} \|w_\epsilon\|_{X^{-\beta}} \leq C \|w_\epsilon\|_{X^{-\beta}}. \end{aligned}$$

Hence, $\{v_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$ is a bounded family in X^α . Moreover,

$$v_\epsilon = A_{-\beta}^{-1}(F_\epsilon)'_\Omega(u^*)v_\epsilon + A_{-\beta}^{-1}w_\epsilon.$$

By compactness of $A_{-\beta}^{-1} : X^{-\beta} \rightarrow X^\alpha$, we get that $\{A_{-\beta}^{-1}w_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$ has a convergent subsequence in X^α . Moreover, using the Lemma 4.10, $\{A_{-\beta}^{-1}(F_\epsilon)'_\Omega(u^*)v_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$ has a convergent subsequence in X^α . Therefore, $\{v_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$ has a convergent subsequence in X^α , that is, the family $\{(A_{-\beta} - (F_\epsilon)'_\Omega(u^*))^{-1} w_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$ has a convergent subsequence in X^α , thus it is relatively compact in X^α .

Now, we take $w_\epsilon \rightarrow w$ in $X^{-\beta}$, as $\epsilon \rightarrow 0$. By continuity of the operator $A_{-\beta}^{-1} : X^{-\beta} \rightarrow X^\alpha$, we have

$$A_{-\beta}^{-1}w_\epsilon \rightarrow A_{-\beta}^{-1}w \quad \text{in } X^\alpha, \quad \text{as } \epsilon \rightarrow 0.$$

Moreover, $\{w_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$ is bounded in $X^{-\beta}$, for some $\epsilon_0 > 0$ sufficiently small, and we have that from the above that $\{v_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$, with $\epsilon_0 > 0$ sufficiently small, has a convergent subsequence, which we again denote by $\{v_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$, with limit $v \in X^\alpha$, that is, $v_\epsilon \rightarrow v$ in X^α , as $\epsilon \rightarrow 0$. From Lemma 4.10 we get

$$A_{-\beta}^{-1}(F_\epsilon)'_\Omega(u^*)v_\epsilon \rightarrow A_{-\beta}^{-1}(F_0)'_\gamma(u^*)v \quad \text{in } X^\alpha, \quad \text{as } \epsilon \rightarrow 0.$$

Thus, v satisfies $v = A_{-\beta}^{-1}(F_0)'_\gamma(u^*)v + A_{-\beta}^{-1}w$, and so $v = (A_{-\beta} - (F_0)'_\gamma(u^*))^{-1}w$. Therefore,

$$(A_{-\beta} - (F_\epsilon)'_\Omega(u^*))^{-1}w_\epsilon \rightarrow (A_{-\beta} - (F_0)'_\gamma(u^*))^{-1}w \quad \text{in } X^\alpha, \quad \text{as } \epsilon \rightarrow 0.$$

The limit above is independent of the subsequence, thus whole family $\{(A_{-\beta} - (F_\epsilon)'_\Omega(u^*))^{-1}w_\epsilon\}_{\epsilon \in (0, \epsilon_0]}$ converges to $(A_{-\beta} - (F_0)'_\gamma(u^*))^{-1}w$ in X^α , as $\epsilon \rightarrow 0$. □

Theorem 4.12. *Suppose that (H2), (H3) and (H4) hold, that α, β and p satisfy (2.4) and that u_0^* is a solution of (4.19) which satisfies $0 \notin \sigma(A_{-\beta} - (F_0)'_\gamma(u_0^*))$. Then, there exist $\epsilon_0 > 0$ and $\delta > 0$ such that, for each $0 < \epsilon \leq \epsilon_0$, the equation (4.18) has exactly one solution, u_ϵ^* , in $\{v_\epsilon \in X^\alpha : \|v_\epsilon - u_0^*\|_{X^\alpha} \leq \delta\}$. Furthermore,*

$$u_\epsilon^* \rightarrow u_0^* \quad \text{in } X^\alpha, \quad \text{as } \epsilon \rightarrow 0.$$

In particular, the family of equilibria $\{u_\epsilon^\}_{\epsilon \in [0, \epsilon_0]}$ is lower semicontinuous at $\epsilon = 0$.*

Proof. Since $u_0^* \in \mathcal{E}_0 \subset \mathcal{A}_0$ then by Lemma 3.9, there exists $R > 0$ such that $\|u_0^*\|_{X^\alpha} \leq R$.

Initially, using the Lemma 4.11, we have that there exist $\epsilon_0 > 0$ and $C > 0$ independent of ϵ such that $0 \notin \sigma(A_{-\beta} - (F_\epsilon)'_\Omega(u_0^*))$ and

$$\|(A_{-\beta} - (F_\epsilon)'_\Omega(u_0^*))^{-1}\|_{\mathcal{L}(X^{-\beta}, X^\alpha)} \leq C, \quad \forall \epsilon \in (0, \epsilon_0]. \quad (4.26)$$

Since $(F_\epsilon)_\Omega$ is Fréchet differentiable, uniformly in ϵ , then there exists $\tilde{\delta} > 0$ independent of ϵ such that

$$\begin{aligned} C \|(F_\epsilon)_\Omega(u_\epsilon) - (F_\epsilon)_\Omega(v_\epsilon) - (F_\epsilon)'_\Omega(u_0^*)(u_\epsilon - v_\epsilon)\|_{X^{-\beta}} \\ \leq \frac{1}{2} \|u_\epsilon - v_\epsilon\|_{X^\alpha}, \quad (4.27) \end{aligned}$$

$\forall \epsilon \in (0, \epsilon_0]$, for $\|u_\epsilon - v_\epsilon\| \leq \tilde{\delta}$.

We note that u_ϵ , $0 < \epsilon \leq \epsilon_0$, is a solution of (4.18) if and only if u_ϵ is a fixed point of the map

$$\begin{aligned} \Phi_\epsilon : X^\alpha &\longrightarrow X^\alpha \\ u_\epsilon &\longmapsto \Phi_\epsilon(u_\epsilon) = (A_{-\beta} - (F_\epsilon)'_\Omega(u_0^*))^{-1} ((F_\epsilon)_\Omega(u_\epsilon) - (F_\epsilon)'_\Omega(u_0^*)u_\epsilon). \end{aligned}$$

Initially, we affirm that

$$\Phi_\epsilon(u_0^*) \rightarrow u_0^* \quad \text{in } X^\alpha, \quad \text{as } \epsilon \rightarrow 0. \quad (4.28)$$

In fact, using (4.26), for $0 < \epsilon \leq \epsilon_0$, we have

$$\begin{aligned} &\|\Phi_\epsilon(u_0^*) - u_0^*\|_{X^\alpha} \\ &\leq \|(A_{-\beta} - (F_\epsilon)'_\Omega(u_0^*))^{-1} [(F_\epsilon)_\Omega(u_0^*) - (F_\epsilon)'_\Omega(u_0^*)u_0^* \\ &\quad - ((F_0)_\gamma(u_0^*) - (F_0)'_\gamma(u_0^*)u_0^*)]\|_{X^\alpha} + \\ &\quad + \|[(A_{-\beta} - (F_\epsilon)'_\Omega(u_0^*))^{-1} - (A_{-\beta} - (F_0)'_\gamma(u_0^*))^{-1}] \\ &\quad \quad ((F_0)_\gamma(u_0^*) - (F_0)'_\gamma(u_0^*)u_0^*)\|_{X^\alpha} \\ &\leq C \left(\|(F_\epsilon)_\Omega(u_0^*) - (F_0)_\gamma(u_0^*)\|_{X^{-\beta}} + \|(F_\epsilon)'_\Omega(u_0^*)u_0^* - (F_0)'_\gamma(u_0^*)u_0^*\|_{X^{-\beta}} \right) \\ &\quad + \|[(A_{-\beta} - (F_\epsilon)'_\Omega(u_0^*))^{-1} - (A_{-\beta} - (F_0)'_\gamma(u_0^*))^{-1}] \\ &\quad \quad ((F_0)_\gamma(u_0^*) - (F_0)'_\gamma(u_0^*)u_0^*)\|_{X^\alpha} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0. \end{aligned}$$

This follows from item 3 of Lemma 4.2, item 5 of Lemma 4.3 and Lemma 4.11.

Next, we show that, for each $0 < \epsilon \leq \epsilon_0$, for some $\epsilon_0 > 0$ sufficiently small, Φ_ϵ is a contraction map from the closed ball $\bar{B}_\delta(u_0^*) = \{v_\epsilon \in X^\alpha : \|v_\epsilon - u_0^*\|_{X^\alpha} \leq \delta\}$ into itself, where $\delta = \frac{\tilde{\delta}}{2}$. First, we show that Φ_ϵ is a contraction on the $\bar{B}_\delta(u_0^*)$ (uniformly in ϵ). Let $u_\epsilon, v_\epsilon \in \bar{B}_\delta(u_0^*)$ and using (4.26) and (4.27), for $0 < \epsilon \leq \epsilon_0$, we have

$$\begin{aligned} &\|\Phi_\epsilon(u_\epsilon) - \Phi_\epsilon(v_\epsilon)\|_{X^\alpha} = \\ &\|(A_{-\beta} - (F_\epsilon)'_\Omega(u_0^*))^{-1} [(F_\epsilon)_\Omega(u_\epsilon) - (F_\epsilon)_\Omega(v_\epsilon) - (F_\epsilon)'_\Omega(u_0^*)(u_\epsilon - v_\epsilon)]\|_{X^\alpha} \\ &\leq C \|(F_\epsilon)_\Omega(u_\epsilon) - (F_\epsilon)_\Omega(v_\epsilon) - (F_\epsilon)'_\Omega(u_0^*)(u_\epsilon - v_\epsilon)\|_{X^{-\beta}} \leq \frac{1}{2} \|u_\epsilon - v_\epsilon\|_{X^\alpha}. \end{aligned}$$

To show that Φ_ϵ maps $\bar{B}_\delta(u_0^*)$ into itself, we observe that if $u_\epsilon \in \bar{B}_\delta(u_0^*)$, then

$$\begin{aligned} \|\Phi_\epsilon(u_\epsilon) - u_0^*\|_{X^\alpha} &\leq \|\Phi_\epsilon(u_\epsilon) - \Phi_\epsilon(u_0^*)\|_{X^\alpha} + \|\Phi_\epsilon(u_0^*) - u_0^*\|_{X^\alpha} \\ &\leq \frac{\delta}{2} + \|\Phi_\epsilon(u_0^*) - u_0^*\|_{X^\alpha}, \quad \text{for } \epsilon \in (0, \epsilon_0]. \end{aligned}$$

By convergence in (4.28), we have that there exists $\epsilon_0 > 0$ such that

$$\|\Phi_\epsilon(u_\epsilon) - u_0^*\|_{X^\alpha} \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta, \quad \text{for } \epsilon \in (0, \epsilon_0].$$

Hence, $\Phi_\epsilon : \bar{B}_\delta(u_0^*) \rightarrow \bar{B}_\delta(u_0^*)$ is a contraction for all $0 < \epsilon \leq \epsilon_0$. By Contraction Theorem follows that, for each $0 < \epsilon \leq \epsilon_0$, Φ_ϵ has a unique fixed point, u_ϵ^* , in $\bar{B}_\delta(u_0^*)$.

To show that $u_\epsilon^* \rightarrow u_0^*$ in X^α , as $\epsilon \rightarrow 0$, we proceed in the following manner: since Φ_ϵ is a contraction map from $\bar{B}_\delta(u_0^*)$ into itself, then

$$\begin{aligned} \|u_\epsilon^* - u_0^*\|_{X^\alpha} &= \|\Phi_\epsilon(u_\epsilon^*) - u_0^*\|_{X^\alpha} \\ &\leq \|\Phi_\epsilon(u_\epsilon^*) - \Phi_\epsilon(u_0^*)\|_{X^\alpha} + \|\Phi_\epsilon(u_0^*) - u_0^*\|_{X^\alpha} \\ &\leq \frac{1}{2} \|u_\epsilon^* - u_0^*\|_{X^\alpha} + \|\Phi_\epsilon(u_0^*) - u_0^*\|_{X^\alpha}. \end{aligned}$$

Thus, using (4.28),

$$\|u_\epsilon^* - u_0^*\|_{X^\alpha} \leq 2 \|\Phi_\epsilon(u_0^*) - u_0^*\|_{X^\alpha} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

Hence and by compactness of \mathcal{E}_0 (Lemma 4.5), we have that the family of equilibria $\{\mathcal{E}_\epsilon\}_{\epsilon \in [0, \epsilon_0]}$ is lower semicontinuity at $\epsilon = 0$. \square

Remark 4.13. *The Theorem 4.12 shows more than continuity of the set equilibria, since it shows that if u_0^* is a solution of the equation (4.19), which satisfies $0 \notin \sigma(A_{-\beta} - (F_0)'_\gamma(u_0^*))$, then, for each $0 < \epsilon \leq \epsilon_0$, with ϵ_0 sufficiently small, there exists a unique solution u_ϵ^* of (4.18) in a neighborhood of u_0^* .*

Corollary 4.14. *Suppose that (H2), (H3) and (H4) hold, that α , β and p satisfy (2.4) and that u_0^* is a hyperbolic solution of (4.19). Then, there exist $\epsilon_0 > 0$ and $\delta > 0$ such that, for each $0 < \epsilon \leq \epsilon_0$, the equation (4.18) has exactly one solution, u_ϵ^* , in $\{v_\epsilon \in X^\alpha : \|v_\epsilon - u_0^*\|_{X^\alpha} \leq \delta\}$. Moreover,*

$$u_\epsilon^* \rightarrow u_0^* \quad \text{in } X^\alpha, \quad \text{as } \epsilon \rightarrow 0.$$

Theorem 4.15. *Suppose that (H2), (H3) and (H4) hold and that α , β and p satisfy (2.4). If all solutions u_0^* of (4.19) satisfy $0 \notin \sigma(A_{-\beta} - (F_0)'_\gamma(u_0^*))$, then (4.19) has a finite number m of solutions, $u_{0,1}^*, \dots, u_{0,m}^*$, and there*

exists $\epsilon_0 > 0$ such that, for each $0 < \epsilon \leq \epsilon_0$, the equation (4.18) has exactly m solutions, $u_{\epsilon,1}^*, \dots, u_{\epsilon,m}^*$. Moreover, for all $i = 1, \dots, m$,

$$u_{\epsilon,i}^* \rightarrow u_{0,i}^* \quad \text{in } X^\alpha, \quad \text{as } \epsilon \rightarrow 0.$$

Proof. The proof follows of Proposition 4.9 and Theorem 4.12. \square

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