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On semiperfect rings of injective dimension one

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Abstract. We give a characterization of right Noetherian semiprime semiperfect and semidistributive rings with $inj.\,dim_A A_A \leqslant 1$.

1. Introduction

Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ be the field of rational numbers and $p \in \mathbb{Z}$ be a prime. Denote by $\mathbb{Z}_p$ the following ring:

$$\mathbb{Z}_p = \left\{ \frac{m}{n} \in \mathbb{Q} \,|\, (n,p) = 1 \right\}.$$

Obviously, every nonzero proper ideal J in $\mathbb{Z}_p$ is principal and has the form $p^k\mathbb{Z}_p$ for some positive k. So, $\mathbb{Z}_p$ is the principal ideal domain and all its ideals form the following descending chain:

$$\mathbb{Z}_p \supset p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset \ldots \supset p^k\mathbb{Z}_p \supset \ldots .$$

Clearly, $\bigcap\limits_{k=1}^{\infty} p^k\mathbb{Z}_p = 0$.

We have the following exact sequence

$$0 \to \mathbb{Z}_p \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}_p \to 0.$$

It is well-known that the $\mathbb{Z}_p$-modules $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}_p$ are injective (recall that $\mathbb{Q}/\mathbb{Z}_p$ is the abelian group p^∞). So, $inj.\,dim_{\mathbb{Z}_p} \mathbb{Z}_p = 1$.

There are many papers devoted to study of injective dimension of rings (see, for example, [1], [2], [9], [3], [4]).

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In the present paper we give a description of right Noetherian semiprime semiperfect and semidistributive rings with the injective dimension at most one.

We will use the results and terminology of [11]. All rings are associative with nonzero identity. A ring A is decomposable if $A = A_1 \times A_2$, otherwise A is indecomposable.

Recall that a module M is called **distributive** if for all submodules K, L, N

$$K \cap (L + N) = K \cap L + K \cap N.$$

Clearly, a submodule and a quotient module of a distributive module are distributive. A module is called **semidistributive** if it is a direct sum of distributive modules. A ring A is called **right** (**left**) **semidistributive** if the right (left) regular module A_A ($_AA$) is semidistributive. A right and left semidistributive ring is called **semidistributive**.

Obviously, every uniserial module is distributive and every serial module is semidistributive.

Theorem 1.1. [11, Theorem 14.1.6] *A semiprimary right semidistributive ring A is right Artinian.*

Definition 1.2. *The endomorphism ring of an indecomposable projective module over a semiperfect ring is called a* **principal** *endomorphism ring.*

The following is a decomposition theorem for semiprime semiperfect rings.

Theorem 1.3. [11, Theorem 14.4.3] *A semiprime semiperfect ring is a finite direct product of indecomposable rings. An indecomposable semiprime semiperfect ring is either a simple Artinian ring or an indecomposable semiprime semiperfect ring such that all its principal endomorphism rings are non-Artinian.*

We write $SPSD$-ring A for a semiperfect and semidistributive ring A.

Definition 1.4. *A ring A is called* **semimaximal** *if it is a semiperfect semiprime right Noetherian ring such that for each local idempotent $e \in A$ the ring eAe is a discrete valuation ring (not necessarily commutative), i.e., all principal endomorphism rings of A are discrete valuation rings.*

Proposition 1.5. [11, Proposition 14.4.12] *A semimaximal ring is a finite direct product of prime semimaximal rings.*

Theorem 1.6. [11, Theorem 14.5.1] *The following conditions for a semiperfect semiprime right Noetherian ring A are equivalent:*

(a) *A is semidistributive;*

(b) *A is a direct product of a semisimple Artinian ring and a semimaximal ring.*

Theorem 1.7. [11, Theorem 14.5.2] *Each semimaximal ring is isomorphic to a finite direct product of prime rings of the following form:*

$$A = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \dots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \dots & \pi^{\alpha_{2n}}\mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \dots & \mathcal{O} \end{pmatrix}, \qquad (1)$$

where $n \geqslant 1$, $\mathcal{O}$ is a discrete valuation ring with a prime element π, and the α_{ij} are integers such that $\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}$ for all i, j, k ($\alpha_{ii} = 0$ for any i).

A ring A is called a **tiled order** if it is a prime Noetherian $SPSD$-ring with nonzero Jacobson radical. Every tiled order is isomorphic to a ring of the form (1).

Denote by $M_n(B)$ the ring of all $n \times n$-matrices over a ring B.

Throughout of this paper, unless specifically noted, A denotes a tiled order with the classical ring of fractions $Q = M_n(D)$, where D is the classical division ring of fractions of $\mathcal{O}$.

Definition 1.8. *An integer matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ is called*

- *an* **exponent** *matrix if $\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}$ and $\alpha_{ii} = 0$ for all i, j, k;*
- *a* **reduced** *exponent matrix if $\alpha_{ij} + \alpha_{ji} > 0$ for all $i, j, i \neq j$.*

We use the following notation: $A = \{\mathcal{O}, \mathcal{E}(A)\}$, where $\mathcal{E}(A) = (\alpha_{ij})$ is the exponent matrix of the ring A, i.e.,

$$A = \sum_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O},$$

in which e_{ij} are the matrix units. If a tiled order is reduced, i.e. $A/R(A)$ is the direct product of division rings, then $\alpha_{ij} + \alpha_{ji} > 0$ if $i \neq j$, i.e. $\mathcal{E}(A)$ is reduced. As usually, $R(A)$ denotes the Jacobson radical of A.

It is well-known that every semiperfect ring is Morita equivalent to a reduced semiperfect ring. Let M be a right A-module. Denote the injective dimension of M by $inj.dim\, _AM$. Let A_A ($_AA$) be the right (left) regular A-module, i.e. A_A is the right module over itself. Obviously, if A and B are Morita equivalent semiperfect rings, then $inj.dim_A A_A = inj.dim_B B_B$.

Hence, every right Noetherian semiprime $SPSD$-ring A is Morita equivalent to $A_1 \times \ldots \times A_m$, where A_i is either a division ring D_i, or $A_i =$

$\{\mathcal{O}_i, \mathcal{E}(A_i)\}$, here $\mathcal{E}(A_i)$ is the reduced exponent matrix. Note that $inj.\,dim\,_{D_i}D_i$ equals zero.

Definition 1.9. *A reduced exponent matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ is called* **Gorenstein** *if there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i, k = 1, \ldots, n$.*

The main result of this paper is the following theorem

Main Theorem. *Let $A = \{\mathcal{O}, \mathcal{E}(A)\}$ be a reduced prime Noetherian SPSD-ring with exponent matrix $\mathcal{E}(A) = (\alpha_{ij}) \in M_n(\mathbb{Z})$. Then $inj.\,dim_A A_A = 1$ if and only if the matrix $\mathcal{E}(A)$ is Gorenstein. In this case $inj.dim_{_AA}A = 1$.*

2. Tiled orders over discrete valuation rings and exponent matrices

Exponent matrices appear in the theory of tiled orders over a discrete valuation ring. Many properties of such orders and their quivers are completely determined by its exponent matrices.

Definition 2.1. *Let A be a tiled order. A* **right (left) A-lattice** *is a right (left) A-module which is a finitely generated free $\mathcal{O}$-module.*

In particular, all finitely generated projective A-modules are A-lattices.

We shall denote by $Lat_r(A)$ (resp. $Lat_l(A)$) the category of right (resp. left) A-lattices.

Among all A-lattices we single out the so-called **irreducible** A-lattices, i.e., A-lattices contained in a simple right (resp. left) Q-module U (resp. V). These lattices form a poset $S_r(A)$ (resp. $S_l(A)$) with respect to inclusion. As it was shown in [11, Section 14.5], any right (resp. left) irreducible A-lattice M (resp. N) lying in U (resp. in V) is an A-module with $\mathcal{O}$-basis $(\pi^{\alpha_1}e_1, \ldots, \pi^{\alpha_n}e_n)$, with

$$\begin{cases} \alpha_i + \alpha_{ij} \geqslant \alpha_j, & \text{if } (\alpha_1, \ldots, \alpha_n) \in S_r(\Lambda) \\ \alpha_j + \alpha_{ij} \geqslant \alpha_i, & \text{if } (\alpha_1, \ldots, \alpha_n)^T \in S_l(\Lambda), \end{cases} \tag{2}$$

where T stands for the transposition operation.

For our purposes, it suffices to consider a reduced tiled order A. In this case, the elements of $S_r(A)$ ($S_l(A)$) are in a bijective correspondence with the integer-valued row vectors $\vec{a} = (\alpha_1, \ldots, \alpha_n)$ (column vectors $\vec{a}^T = (\alpha_1, \ldots, \alpha_n)^T$), where $\vec{a}$ and $\vec{a}^T$ satisfy the conditions (2). We shall write $[M] = (\alpha_1, \ldots, \alpha_n)$ or $M = (\alpha_1, \ldots, \alpha_n)$, if $M \in S_r(A)$.

Let $\vec{b} = (\beta_1, \ldots, \beta_n)$. The order relation $\vec{a} \preceq \vec{b}$ in $S_r(A)$ is defined as follow:

$$\vec{a} \preceq \vec{b} \iff \alpha_i \geqslant \beta_i \text{ for } i = 1, \ldots, n.$$

Since A is a semidistributive ring, $S_r(A)$ and $S_l(A)$ are distributive lattices with respect to addition and intersection.

Proposition 2.2. *There exists only a finite number of irreducible A-lattices up to an isomorphism.*

Proof. Let $A = \{\mathcal{O}, \mathcal{E}(A)\}$ be a tiled order with an exponent matrix $\mathcal{E}(A) = (\alpha_{ij})$. We can assume that $\alpha_{ij} \geqslant 0$ for $i, j = 1, \ldots, n$. Let $M = (\alpha_1, \ldots, \alpha_n) \in S_r(A)$. Considering an isomorphic module we can assume that all $\alpha_1, \ldots, \alpha_n$ are positive integers. Denote $a = min\,(\alpha_1, \ldots, \alpha_n)$. Then $M_1 = (\alpha_1 - a, \ldots, \alpha_n - a)$ is an irreducible A-lattice and $M_1 \simeq M$. Suppose that $\alpha_i = a$. Then $M_1 = (\beta_1, \ldots, \beta_n)$, where $\beta_1, \ldots, \beta_n$ are non-negative and $\beta_i = 0$. Consequently, every irreducible A-lattice M is isomorphic to the lattice M_1 with at least one zero coordinate. We obtain from (2) that $0 \leqslant \beta_j \leqslant \alpha_{ij}$. So, number of irreducible A-lattices of the form M_1 is finite. □

Using the properties of projective covers of finitely generated modules over semiperfect rings, one can characterize projective modules of $S_r(A)$ (resp. $S_l(A)$) in the following way:

Proposition 2.3. *An irreducible A-lattice is projective if and only if it contains exactly one maximal submodule.*

Definition 2.4. *Two exponent matrices $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$ are called* **equivalent** *if they can be obtained from each other by transformations of the following two types :*

(1) *subtracting an integer α from the entries of the l-th row with simultaneous adding α to the entries of the l-th column;*

(2) *simultaneous interchanging of two rows and the same numbered columns.*

Let $A = \sum\limits_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O}$ and $B = \sum\limits_{i,j=1}^{n} e_{ij}\pi^{\beta_{ij}}\mathcal{O}$ be tiled orders, e_{ij} are the matrix units, i.e., $A = \{\mathcal{O}, \mathcal{E}(A)\}$ and $B = \{\mathcal{O}, \mathcal{E}(B)\}$. Obviously, if $\mathcal{E}(A)$ and $\mathcal{E}(B)$ are equivalent, then A and B are isomorphic.

Proposition 2.5. *Suppose $\mathcal{E} = (\alpha_{ij})$, $\Theta = (\theta_{ij})$ are exponent matrices, and Θ is obtained from $\mathcal{E}$ by a transformation of type (1). Then $[Q(\mathcal{E})] = [Q(\Theta)]$. If $\mathcal{E}$ is a reduced Gorenstein exponent matrix with permutation $\sigma(\mathcal{E})$, then Θ is also reduced Gorenstein with $\sigma(\Theta) = \sigma(\mathcal{E})$.*

Proof. We have

$$\theta_{ij} = \begin{cases} \alpha_{ij}, & \text{if } i \neq l, j \neq l, \\ 0, & \text{if } i = l, j = l, \\ \alpha_{lj} - t, & \text{if } i = l, j \neq l, \\ \alpha_{il} + t, & \text{if } i \neq l, j = l, \end{cases}$$

where t is an integer. It can be directly checked that if $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$ for some i, j, k, then $\theta_{ij} + \theta_{jk} = \theta_{ik}$. Since these transformations are invertible, the inverse transformations have a similar form. So the equality $\theta_{ij} + \theta_{jk} = \theta_{ik}$ implies $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$. Therefore, $\theta_{ij} + \theta_{jk} = \theta_{ik}$ if and only if $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$.

Denote $\Theta^{(1)} = (\mu_{ij})$ and $\Theta^{(2)} = (\nu_{ij})$.

The equalities $\gamma_{ij} = \beta_{ij}$, $\nu_{ij} = \mu_{ij}$ or inequalities $\gamma_{ij} > \beta_{ij}$, $\nu_{ij} > \mu_{ij}$ hold simultaneously for the entries of the matrices $(\beta_{ij}) = \mathcal{E}_1$, $(\mu_{ij}) = \Theta^{(1)}$, $(\gamma_{ij}) = \mathcal{E}^{(2)}$, $(\nu_{ij}) = \Theta^{(2)}$. Therefore, $\mathcal{E}^{(2)} - \mathcal{E}^{(1)} = \Theta^{(2)} - \Theta^{(1)}$ and $[Q(\mathcal{E})] = [Q(\Theta)]$.

Suppose that $\mathcal{E}$ is a reduced Gorenstein exponent matrix with permutation $\sigma(\mathcal{E})$, i. e., $\alpha_{ij} + \alpha_{j\sigma(i)} = \alpha_{i\sigma(i)}$. Whence, $\theta_{ij} + \theta_{j\sigma(i)} = \theta_{i\sigma(i)}$. This means that the matrix Θ is also Gorenstein with the same permutation $\sigma(\mathcal{E})$. □

Let τ be a permutation which determines simultaneous transpositions of rows and columns of the reduced exponent matrix $\mathcal{E}$ under transformations of the second type. Then $\theta_{ij} = \alpha_{\tau(i)\tau(j)}$ and $\Theta = P_\tau^T \mathcal{E} P_\tau$, where $P_\tau = \sum\limits_{i=1}^{n} e_{i\tau(i)}$ is the permutation matrix, and P_τ^T stands for the transposed matrix of P_τ. Since $\alpha_{ij} + \alpha_{j\sigma(i)} = \alpha_{i\sigma(i)}$ and $\alpha_{ij} = \theta_{\tau^{-1}(i)\tau^{-1}(j)}$, we have $\theta_{\tau^{-1}(i)k} + \theta_{k\tau^{-1}(\sigma(i))} = \theta_{\tau^{-1}(i)\tau^{-1}(\sigma(i))}$. Hence the permutation π of Θ satisfies $\pi(\tau^{-1}(i)) = \tau^{-1}(\sigma(i))$ for all i. Whence, $\pi = \tau^{-1}\sigma\tau$.

Since

$$\mu_{ij} = \beta_{\tau(i)\tau(j)}, \quad \nu_{ij} = \min_k(\mu_{ik} + \mu_{kj}) = \min_l(\beta_{\tau(i)l} + \beta_{l\tau(j)}) = \gamma_{\tau(i)\tau(j)},$$

it follows that,

$$\tilde{q}_{ij} = \nu_{ij} - \mu_{ij} = \gamma_{\tau(i)\tau(j)} - \beta_{\tau(i)\tau(j)} = q_{\tau(i)\tau(j)},$$

where $[\tilde{Q}] = (\tilde{q}_{ij})$ is the adjacency matrix of the quiver $\tilde{Q}$ of Θ. So we proved the following statement.

Proposition 2.6. *Under the transformations of the second type the adjacency matrix $[\tilde{Q}]$ of $Q(\Theta)$ is changed according to the formula: $[\tilde{Q}] =$*

$P_\tau^T[Q]P_\tau$, where $[Q] = [Q(\mathcal{E})]$. If $\mathcal{E}$ is Gorenstein, then Θ is also Gorenstein, and for the new permutation π we have: $\pi = \tau^{-1}\sigma\tau$, i.e., $\sigma(\Theta) = \tau^{-1}\sigma(\mathcal{E})\tau$.

Note that the type of a permutation does not change under transformations of the second type. Therefore, in order to describe the reduced Gorenstein exponent matrices, one needs to examine matrices with different types of permutations. Further, to simplify calculations we can assume that a row or a column of $\mathcal{E}$ is zero. This can be always obtained by transformations of the first type, moreover the entries of a new exponent matrix will be non-negative integers. Indeed, let $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ be an exponent matrix. Subtracting α_{1i} from the entries of the i-th column and adding α_{1i} to the entries of the i-th row, we obtain a new exponent matrix

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \theta_{21} & 0 & \theta_{23} & \dots & \theta_{2n} \\ \theta_{31} & \theta_{32} & 0 & \dots & \theta_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_{n1} & \theta_{n2} & \theta_{n3} & \dots & 0 \end{pmatrix}.$$

The first row of Θ equals zero. Consequently, $\theta_{1i}+\theta_{ij} \geqslant \theta_{1j} = 0$ and $\theta_{ij} \geqslant 0$ for $i, j = 1, \dots, n$.

3. Duality in Noetherian rings

We use the duality in Noetherian rings following H.Bass, J.Dieudonne, J.Jans, and K.Morita.

Let M be a right A-module and let

$$M^* = Hom_A(M, A_A). \tag{3}$$

Obviously, it is an additive group and it can be considered as a left A-module if we define $a\varphi$ by the formula $(a\varphi)(m) = \varphi(ma)$, where $a \in A$, $\varphi \in M^*$, $m \in M$. This left A-module is called **dual** to the right A-module M. Analogously, for any left A-module N we can define the dual module

$$N^* = Hom_A(N, {}_AA),$$

which is a right A-module, if we set $(\psi a)(x) = \psi(x)a$ for $a \in A$, $\psi \in N^*$, $x \in N$. Obviously, isomorphic modules have isomorphic duals.

Let $f : N \to M$ be a homomorphism of right A-modules. Then we may define a map $f^* : M^* \to N^*$ by the formula $f^*(\varphi) = \varphi f$ for $\varphi \in M^*$. It is easy to show that f^* is an A-homomorphism of left A-modules. This homomorphism f^* is called **dual** to f.

Let F be a free A-module with a finite free basis $f_1, f_2, \dots, f_n$. Define an A-homomorphism $\varphi_i : F \to A$ by $\varphi_i(f_j) = \delta_{ij}$ for $i, j = 1, 2, \dots, n$, where

δ_{ij} is the Kronecker delta. Then $\varphi_i \in F^*$. It is easy to show that F^* is a free A-module with a free basis $\varphi_1, \dots, \varphi_n$. This basis is called **dual** to $f_1, f_2, \dots, f_n$.

Lemma 3.1. *Let P be a finitely generated projective module. Then the dual module P^* is also a finitely generated projective A-module.*

Proof. Suppose that P is generated by elements $x_1, \dots x_n$ and let F be a free module with a free basis $f_1, f_2, \dots, f_n$. Then there is an epimorphism $\pi : F \to P$ with $\pi(f_i) = x_i$ for $i = 1, \dots, n$. Since P is projective, there is a homomorphism $\sigma : P \to F$ such that $\pi\sigma = 1_P$. Consequently, $\sigma^*\pi^* = (\pi\sigma)^* = 1_{P^*}$. Therefore P^* is a direct summand of a free module F^* which is free with the finite basis of n elements. So P^* is a finitely generated projective module. □

Lemma 3.2. *Let A be a right Noetherian ring. Then the dual to any finitely generated left A-module is also finitely generated.*

Proof. Let M be a finitely generated left A-module. Then there is an exact sequence $0 \to N \to F \to M \to 0$ with a free module F with a finite base. Applying the duality functor $Hom_A(*, A)$ we obtain that M^* is a submodule of F^*. Since F^* is a free right A-module with a finite basis and A is a right Noetherian ring, then M^* is also a finitely generated A-module, by [11, Corollary 3.1.13]. □

Let M be a right A-module with a dual module M^*. Then M^* itself has a dual module M^{**}. Suppose $m \in M$ and $f \in M^* = Hom_A(M, A)$. Define a map

$$\varphi_m : M^* \to A$$

by $\varphi_m(f) = f(m)$. Obviously,

$$\varphi_m(f_1 + f_2) = \varphi_m(f_1) + \varphi_m(f_2).$$

For any $a \in A$ we have $\varphi_m(af) = af(m) = a\varphi_m(f)$. Thus φ_m is an A-homomorphism, i.e., $\varphi_m \in M^{**}$. Consider the map

$$\delta_M : M \to M^{**} \tag{4}$$

defined by $\delta_M(m)(f) = f(m)$ for $m \in M$ and $f \in M^*$. It is easy to verify that δ_M is an A-homomorphism.

Definition 3.3. *A module M is called* **reflexive** *if δ_M is an isomorphism. It is called* **semi-reflexive** *if δ_M is a monomorphism.*

Note that any finite dimensional vector space is reflexive.

Lemma 3.4. *Any submodule of a semi-reflexive module is semi-reflexive and any direct summand of a reflexive module is reflexive.*

Proof. Suppose that M is a semi-reflexive A-module and N is a submodule of M. Let $i : N \to M$ be an inclusion mapping. Then the following diagram

$$\begin{array}{ccc} N & \xrightarrow{\delta_N} & N^{**} \\ \downarrow i & & \downarrow i^{**} \\ M & \xrightarrow{\delta_M} & M^{**} \end{array}$$

is commutative. Since i and δ_M are monomorphisms, δ_N is also a monomorphism. Therefore N is semi-reflexive.

Let N be a direct summand of a reflexive module M. Then there are an inclusion map $i : N \to M$ and an epimorphism $\pi : M \to N$ such that $\pi i = 1_N$. Then $\pi^{**} i^{**} = (\pi i)^{**} = 1_{N^{**}}$ is an epimorphism. The diagram

$$\begin{array}{ccc} M & \xrightarrow{\delta_M} & M^{**} \\ \downarrow \pi & & \downarrow \pi^{**} \\ M & \xrightarrow{\delta_N} & N^{**} \end{array}$$

is commutative. Since δ_M is an isomorphism and π^{**} is an epimorphism, δ_N is also an epimorphism. But from the first part of this lemma δ_N is a monomorphism. So δ_N is an isomorphism, i.e., N is reflexive. □

Proposition 3.5. *Each finitely generated projective module is reflexive. In particular, a free module with a finite free basis is reflexive.*

Proof. Let F be a free module with a finite free basis $f_1, f_2, \ldots, f_n$ and let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be the free basis of F^* dual to $f_1, f_2, \ldots, f_n$. Let ψ_1, ψ_2, $\ldots$, ψ_n be a basis of F^{**} dual to $\varphi_1, \varphi_2, \ldots, \varphi_n$. Then $\delta_F(f_i)$ and ψ_i both belong to $Hom_A(F^*, A)$ and

$$\delta_F(f_i)(\varphi_j) = \varphi_j(f_i) = \delta_{ji} = \psi_i(\varphi_j).$$

This implies that $\delta_F(f_i) = \psi_i$ and δ_F is an isomorphism, i.e., F is reflexive.

Let P be a finitely generated projective module. Then P is a direct summand of a free module with a finite basis. Hence P is reflexive, by Lemma 3.4. □

Lemma 3.6. *The dual of an arbitrary module is semi-reflexive. The dual of a reflexive module is reflexive.*

Proof. Let M be an A-module. If we apply the duality functor to the A-homomorphism $\delta_M : M \to M^{**}$ we obtain an A-homomorphism $\delta_M^* : M^{***} \to M^*$. Then it is easy to show that

$$\delta_M^* \delta_{M^*} = 1_{M^*}. \tag{5}$$

It follows from this equality that δ_{M^*} is a monomorphism, so M^* is semi-reflexive. If M is reflexive, then δ_M is an isomorphism, and so is δ_M^*. But then from (5) $\delta_{M^*} = (\delta_M^*)^{-1}$ is also an isomorphism, i.e., M^* is reflexive. $\square$

Lemma 3.7. *Let A be a right Noetherian ring. Then any finitely generated submodule of a free module with a finite base is semi-reflexive.*

Proof. This follows from Proposition 3.5 and Lemma 3.4. $\square$

4. **Duality in tiled orders**

In this section we shall introduce duality in tiled orders and study its properties.

Proposition 4.1. *Let A be a tiled order with its classical ring of fractions Q. Then Q is a flat and injective right and left A-module.*

Proof. The classical ring of fractions Q is the direct limit of flat submodules $\pi^k A = A\pi^k$ of A, for $k \in \mathbf{Z}$. Then Q is flat, by [11, Proposition 5.4.6].

To prove the injectiveness of Q we use Baer's criterion (see [11, Proposition 5.4.6]). Let I be a right ideal in A. Since A is a Noetherian ring, I is a finitely generated ideal. Take the diagram

$$\begin{tikzcd} 0 \arrow[r] & I_A \arrow[r, "i"] \arrow[d, "\varphi"] & A_A \\ & Q & \end{tikzcd} ,$$

where i is a monomorphism. Since Q is flat, the sequence

$$0 \longrightarrow I_A \otimes Q \xrightarrow{i \otimes 1_Q} A_A \otimes Q$$

is exact. Then, by [11, Proposition 5.4.11], we obtain the following diagramm

$$\begin{tikzcd} 0 \arrow[r] & \tilde{I} \arrow[r, "i \otimes 1_Q"] \arrow[d, "\tilde{\varphi}"] & Q \\ & Q & \end{tikzcd} ,$$

where $\tilde{I} \simeq IQ$ and $\tilde{\varphi} = \varphi \otimes 1_Q$. Since $Q = M_n(D)$ is a simple Artinian ring, then Q is a two-sided injective Q-module. Therefore, by Baer's criterion, there is a homomorphism $\tilde{\psi} : Q \to Q$ such that $\tilde{\varphi} = \tilde{\psi}\tilde{i}$. Restricting $\tilde{i}$ and $\tilde{\varphi}$ on I_A, $\tilde{\psi}$ on A_A we obtain $\varphi = \psi i$. Thus Q is an injective A-module. $\square$

Now we shall consider finitely generated semi-reflexive A-modules.

Proposition 4.2. *A finitely generated A-module M is semi-reflexive if and only if M is isomorphic to a submodule of a free A-module of finite rank A^m.*

Proof. If $M \subset A^m$, then M is semi-reflexive, by Lemma 3.4.

Conversely, let M be a finitely generated semi-reflexive A-module. We shall write $X^* = Hom_A(X, A)$ for any A-module X. An epimorphism $A^m \to M \to 0$ induces a monomorphism $0 \to M^* \to (A^m)^*$. But $A^* = Hom_A(A, A) \simeq A$ and M^* is isomorphic to a submodule of A^m. Since A is a Noetherian ring, M^* is a finitely generated A-module and therefore there is an exact sequence $A^r \to M^* \to 0$. Then $0 \to M^{**} \to A^r$ is a monomorphism. Since M is semi-reflexive then $\delta_M : M \to M^{**}$ is a monomorphism. Therefore, M is isomorphic to a submodule of a free A-module of a finite rank. □

Let A be a tiled order of the form (1). Recall that an A-module M is called an A-lattice if it is a finitely generated free $\mathcal{O}$-module (see [11, p.353]).

Proposition 4.3. *Let A be a tiled order. Then an A-module M is finitely generated semi-reflexive if and only if M is an A-lattice.*

Proof. Let $A = \sum\limits_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O} \subset \sum\limits_{i,j=1}^{n} e_{ij}D = Q = M_n(D)$. Denote by E_n the identity matrix of $M_n(D)$. Obviously, $E_n = \sum\limits_{i=1}^{n} e_{ii}$, where e_{ii} are local matrix idempotents of A. Let $X = \{x \in M_n(D) : xe_{ij} = e_{ij}x \text{ for } i, j = 1, \ldots, n\}$ and $Y = \{y \in A : ye_{ij} = e_{ij}y \text{ for } i, j = 1, \ldots, n\}$. Obviously, $X = \{dE_n\}$, where $d \in D$ and $Y = \{\alpha E_n\}$, where $\alpha \in \mathcal{O}$. So we can assume that D is a subring of $M_n(D)$ and $\mathcal{O}$ is a subring of A (D coincides with X and $\mathcal{O}$ coincides with Y). Therefore, A is a free $\mathcal{O}$-module of rank n^2, i.e., A-lattice. By Proposition 4.2, an A-module M is finitely generated semi-reflexive if and only if M is an A-lattice. Obviously, $A \otimes_{\mathcal{O}} D = M_n(D) = Q$ and $M \otimes_A Q = M \otimes_A (A \otimes_{\mathcal{O}} D) = M \otimes_{\mathcal{O}} D$, by [11, Proposition 4.5.3]. In this case $\tilde{M} = M \otimes_{\mathcal{O}} D$ is a finite dimensional vector space over D and M is a complete right A-lattice in $\tilde{M}$, where $rank_{\mathcal{O}} M = dim_D \tilde{M}$. □

Proposition 4.4. *Let*

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{p} N \longrightarrow 0$$

be an exact sequence of right A-modules. If $L, N \in Lat_r(A)$ then $M \in Lat_r(A)$ as well.

Proof. Let $m \neq 0$, $m \in M$ and $m\pi^t E_n = 0$ for some positive $t \in \mathbf{Z}$. Then $p(m)\pi^t E_n = 0$ and $p(m) = 0$. Therefore $m \in Ker\, p = Im\, i$, i.e., $m = i(l)$, where $l \in L$ and $m\pi^t E_n = i(l\pi^t E_n) = 0$. Thus $l\pi^t E_n = 0$. Since $L \in Lat_r(A)$ we obtain $l = 0$ and $m = 0$. $\square$

We shall establish now the duality between the categories $Lat_r(A)$ and $Lat_l(A)$. Let $M \in Lat_r(A)$. Denote $M^{\#} = Hom_{\mathcal{O}}(M, \mathcal{O})$. For any $f \in M^{\#}$ and $a \in A$ we can define af by the formula $(af)(m) = f(ma)$ where $m \in M$. Then it is easy to verify that $M^{\#}$ is a left A-module.

Since $M \in Lat_r(A)$, it is a free $\mathcal{O}$-module with a finite $\mathcal{O}$-basis e_1, e_2, $\ldots$, e_n. We can define an $\mathcal{O}$-homomorphism $\varphi_i : M \to \mathcal{O}$ by the formula $\varphi_i(e_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$, where δ_{ij} is the Kronecker symbol. Then $\varphi_i \in M^{\#}$. It is easy to see that $M^{\#}$ is a free $\mathcal{O}$-module with $\mathcal{O}$-basis $\varphi_1, \ldots, \varphi_n$. This $\mathcal{O}$-basis is called the **dual** $\mathcal{O}$-basis of $M^{\#}$. Thus, $M^{\#} \in Lat_l(A)$. If $M \in Lat_l(A)$, then $M^{\#} \in Lat_r(A)$.

Let $\varphi : M \to N$ be a homomorphism, where $M, N \in Lat_r(A)$, i.e., $\varphi \in Hom_A(M, N)$. Then $\varphi^{\#} : N^{\#} \to M^{\#}$ can be defined by formula $(\varphi^{\#} f)(m) = f\varphi(m)$, where $f \in N^{\#}$, is a homomorphism from $N^{\#}$ to $M^{\#}$, i.e., $\varphi^{\#} \in Hom_A(N^{\#}, M^{\#})$. Obviously, if we have homomorphisms $\psi : L \to M$ and $\varphi : M \to N$, then $(\psi\varphi)^{\#} = \psi^{\#}\varphi^{\#}$ and $1_M^{\#} = 1_{M^{\#}}$. Moreover, for any $M \in Lat_r(A)$ we have $M^{\#\#} = M$ and for any $N \in Lat_l(A)$ it is true $N^{\#\#} = N$. Moreover, for any $\varphi : M \to N$ we have $\varphi^{\#\#} = \varphi$. Clearly, we also have $(M \oplus N)^{\#} = M^{\#} \oplus N^{\#}$.

Proposition 4.5. *Let L be a submodule of M and $L, M/L \in Lat_r(A)$. Let $p : M \to M/L$ be the natural projection. Then $M \in Lat_r(A)$ and M has the following $\mathcal{O}$-basis: $e_1, \ldots, e_s, p^{-1}(n_1), \ldots, p^{-1}(n_t)$, where $e_1, \ldots, e_s$ is an $\mathcal{O}$-basis of L and $n_1, \ldots, n_t$ is an $\mathcal{O}$-basis of M/L.*

Proof. By Proposition 4.4, $M \in Lat_r(A)$. Denote $N = M/L$. Let $e_1\alpha_1 + \ldots + e_s\alpha_s + p^{-1}(n_1)\beta_1 + \ldots + p^{-1}(n_t)\beta_t = 0$. Then $e_1\alpha_1 + \ldots + e_s\alpha_s + p^{-1}(n_1\beta_1 + \ldots + n_t\beta_t) = 0$. Obviously, $p(e_1\alpha_1 + \ldots + e_s\alpha_s + p^{-1}(n_1\beta_1 + \ldots + n_t\beta_t)) = n_1\beta_1 + \ldots + n_t\beta_t = 0$. Thus $\beta_1 = \ldots = \beta_t = 0$ and $e_1\alpha_1 + \ldots + e_s\alpha_s = 0$. We obtain $\alpha_1 = \ldots = \alpha_s$. Let $m \in M$. Then $p(m) = n_1\beta_1 + \ldots + n_t\beta_t$ and $m - p^{-1}(n_1\beta_1 + \ldots + n_t\beta_t) \in Ker\, p$. We obtain that $m - p^{-1}(n_1\beta_1 + \ldots + n_t\beta_t) = e_1\alpha_1 + \ldots + e_s\alpha_s$ and $m = e_1\alpha_1 + \ldots + e_s\alpha_s + p^{-1}(n_1)\beta_1 + \ldots + p^{-1}(n_t)\beta_t$. The proposition is proved. $\square$

Proposition 4.6. *Let $L, M, N = M/L$ be as in the previous proposition. Let*

$$0 \longrightarrow L \longrightarrow M \xrightarrow{p} N \longrightarrow 0$$

be an exact sequence. Then there is a dual $\mathcal{O}$-basis $\varphi_1, \ldots, \varphi_s$, $p^{\#}\Theta_1, \ldots,$ $p^{\#}\Theta_t$ of $M^{\#}$, where $\varphi_1, \ldots, \varphi_s$ is a dual $\mathcal{O}$-basis of $L^{\#}$ and $\Theta_1, \ldots, \Theta_t$ is a dual basis of $N^{\#}$.

Proof. By Proposition 4.5, M has an $\mathcal{O}$-basis $e_1, \ldots, e_s$, $p^{-1}(n_1), \ldots,$ $p^{-1}(n_t)$, where $e_1, \ldots, e_s$ is an $\mathcal{O}$-basis of L and $n_1, \ldots, n_t$ is an $\mathcal{O}$-basis of N. We shall verify that $\varphi_1, \ldots, \varphi_s, p^{\#}\Theta_1, \ldots, p^{\#}\Theta_t$ is a dual $\mathcal{O}$-basis to the $\mathcal{O}$-basis $e_1, \ldots, e_s, p^{-1}(n_1), \ldots, p^{-1}(n_s)$. By definition, $\varphi_i(e_j) = \delta_{ij}$ for $i, j = 1, \ldots, s$. We have $p^{\#}\Theta_i(p^{-1}(n_j)) = \Theta_i p(p^{-1}(n_j)) = \Theta_i(n_j) = \delta_{ij}$ for $i, j = 1, \ldots, t$. □

Corollary 4.7. *Let*

$$0 \longrightarrow L \longrightarrow M \xrightarrow{p} M \longrightarrow 0$$

be an exact sequence as above. Then the sequence

$$0 \longrightarrow N^{\#} \xrightarrow{p^{\#}} M^{\#} \longrightarrow L^{\#} \longrightarrow 0$$

is exact.

Corollary 4.8. $Ext^1_A(N, {}_AA^{\#}) = 0$ *for any $N \in Lat_r(A)$.*

Proof. Let

$$0 \longrightarrow {}_AA^{\#} \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence. By Corollary 4.7, we obtain that

$$0 \longrightarrow N^{\#} \longrightarrow M^{\#} \longrightarrow {}_AA \longrightarrow 0$$

is an exact sequence of left A-lattices. Then from projectivity of ${}_AA$ we have $M^{\#} \simeq A \oplus N^{\#}$. Therefore $M^{\#\#} = M \simeq_A A \oplus N$, i.e., $Ext^1_A(N, {}_AA^{\#}) = 0$. □

It is easy to establish the duality of irreducible and completely decomposable A-lattices.

Let $M \in S_r(A)$ and $M = \sum_{i=1}^{n} e_i \pi^{\alpha_i} \mathcal{O}$. If $\varphi_1, \ldots, \varphi_n$ is the dual $\mathcal{O}$-basis for $e_1, \ldots, e_n$, then $\pi^{-\alpha_1}\varphi_1, \ldots, \pi^{-\alpha_n}\varphi_n$ is the dual $\mathcal{O}$-basis for the $\mathcal{O}$-basis $e_1\pi^{\alpha_1}, \ldots, e_n\pi^{\alpha_n}$. Consequently, if $M = (\alpha_1, \ldots, \alpha_n)$, then $M^{\#} = (-\alpha_1, \ldots, -\alpha_n)$. Using the same formula for $N = (\beta_1, \ldots, \beta_n)^T$, we obtain $N^{\#} = (-\beta_1, \ldots, -\beta_n)$. It is easy to see that

$$(M_1 + M_2)^{\#} = M_1^{\#} \cup M_2^{\#} \text{ and } (M_1 \cup M_2)^{\#} = M_1^{\#} + M_2^{\#}$$

for any $M_1, M_2 \in S_r(A)$. Further, if $M_1 \subset M_2$ are two irreducible A-lattices then $M_2^{\#} \subset M_1^{\#}$. (In this case the lattice M_2 is called an **overmodule** of M_1).

Definition 4.9. *An A-lattice M is said to be* **relatively injective** *if $M \simeq {}_AP^{\#}$, where ${}_AP$ is a finitely generated projective left A-module.*

Definition 4.10. *An A-lattice M is called* **completely decomposable** *if it is a direct sum of irreducible A-lattices.*

Corollary 4.11. *A relatively injective A-lattice M is completely decomposable and any relatively injective indecomposable M has the following form: $M = {}_AP^{\#}$, where ${}_AP$ is an indecomposable projective left A-module.*

Proof. The tiled order

$$A = \sum_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O}$$

is a completely decomposable right A-lattice

$$A_A = e_{11}A \oplus \ldots \oplus e_{nn}A$$

and completely decomposable left A-lattice

$${}_AA = Ae_{11} \oplus \ldots \oplus Ae_{nn}.$$

Every finitely generated left projective A-module ${}_AP$ has the following form: ${}_AP = (Ae_{11})^{m_1} \oplus \ldots \oplus (Ae_{nn})^{m_n}$. Obviously, ${}_AP \in Lat_l(A)$ and

$${}_AP^{\#} = (Ae_{11})^{\#m_1} \oplus \ldots \oplus (Ae_{nn})^{\#m_n}.$$

So, ${}_AP^{\#}$ is a completely decomposable right A-lattice. In particular, M is indecomposable if and only if $M = (Ae_{ii})^{\#}$ for some $i = 1, \ldots, n$. □

In what follows we assume that the tiled order A is reduced. In this case $\mathcal{E}(A)$ is reduced, i.e., $\alpha_{ij} + \alpha_{ji} > 0$ for $i \neq j$. An A-lattice $N \subset M_n(D)$ is said to be **complete** if $N \simeq (\mathcal{O}_{\mathcal{O}})^{n^2}$ as a right $\mathcal{O}$-module. If a complete A-lattice N is a left A-module then

$$N = \sum_{i,j=1}^{n} e_{ij}\pi^{\gamma_{ij}}\mathcal{O}.$$

In this case the matrix (γ_{ij}) is said to be the **exponent matrix** of the A-lattice N and we write it by $\mathcal{E}(N)$. Complete A-lattices which are left A-modules are said to be **fractional ideals** of A. Denote by Δ the completely decomposable lattice $A_A^{\#}$.

Lemma 4.12. *A completely decomposable left A-lattice Δ is a complete right A-lattice, and*

$$\mathcal{E}(\Delta) = \begin{pmatrix} 0 & -\alpha_{21} & \dots & -\alpha_{n1} \\ -\alpha_{12} & 0 & \dots & -\alpha_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{1n} & -\alpha_{2n} & \dots & 0 \end{pmatrix}.$$

Proof. We show that the k-th row $(-\alpha_{1k}, -\alpha_{2k}, \dots, -\alpha_{nk})$ of the matrix $\mathcal{E}(\Delta)$ defines an irreducible right A-lattice. Write $\beta_i = -\alpha_{ik}$. We can rewrite the inequality $\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}$ in the form $-\alpha_{ik} + \alpha_{ij} \geqslant -\alpha_{jk}$, i.e., $\beta_i + \alpha_{ij} \geqslant \beta_j$, which implies the assertion of the lemma. □

Corollary 4.13. *The fractional ideal Δ is a relatively injective right and a relatively injective left A-lattice.*

Proof. The proof follows from the relation ${}_A\Delta^{\#} = A_A$. □

Let A be a reduced tiled order and $R = radA$. Then

$$\mathcal{E}(R) = \begin{pmatrix} 1 & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & 1 & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & 1 \end{pmatrix}$$

and

$$\mathcal{E}({}_AR^{\#}) = \mathcal{E}(R^{\#}_A) = \begin{pmatrix} -1 & -\alpha_{21} & \dots & -\alpha_{n1} \\ -\alpha_{12} & -1 & \dots & -\alpha_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{1n} & -\alpha_{2n} & \dots & -1 \end{pmatrix}.$$

We denote $X = {}_AR^{\#}$.

Lemma 4.14. *For $i = 1, \dots, n$ we have that $e_{ii}X$ (Xe_{ii}) is the unique minimal overmodule of $e_{ii}\Delta$ (Δe_{ii}) and $e_{ii}X/e_{ii}\Delta = U_i$, $Xe_{ii}/\Delta e_{ii} = V_i$, where U_i is a simple right A-module and V_i is a simple left A-module.*

Proof. The proof for the left case follows from the fact that $e_{ii}R$ is the unique maximal submodule of $e_{ii}A$ and from the duality properties and the annihilation lemma. The proof for the right case is just the same. □

Note once more, that $e_{ii}\Delta$ (respectively, Δe_{ii}) are all indecomposable relatively right (respectively, left) injective A-lattices (up to isomorphism) and each $e_{ii}X$ (respectively, Xe_{ii}) is the unique minimal overmodule of $e_{ii}\Delta$ (respectively, Δe_{ii}). Moreover, the notion of an indecomposable relatively injective A-lattice and the notion of an irreducible relatively injective A-lattice coincide.

Let A_1 and A_2 be Morita equivalent tiled orders. Then the relatively injective indecomposable A_1-lattices correspond to relatively injective indecomposable A_2-lattices. Thus, from Lemma 4.14 we have the following lemma

Lemma 4.15. *Every relatively injective irreducible A-lattice Q has only one minimal overmodule. Let Q_1 and Q_2 be relatively injective irreducible A-lattices, and let $X_1 \supset Q_1$ and $X_2 \supset Q_2$ be the unique minimal overmodules of Q_1 and Q_2, respectively. Then the simple A-modules X_1/Q_1 and X_2/Q_2 are isomorphic if and only if $Q_1 \simeq Q_2$.*

Next we state the dual statement to Proposition 2.3 the proof of which can be simply obtained from duality properties:

Proposition 4.16. *An irreducible A-lattice is relatively injective if and only if it has exactly one minimal overmodule.*

Now we give an interesting fact about the injective dimension of the lattice ${}_AA^{\#}$.

Proposition 4.17. *Let A be a tiled order. Then $inj.dim_A\,({}_AA^{\#}) = 1$.*

Proof. Let $\mathcal{I}$ be a right ideal of A. Consider the exact sequence $0 \to \mathcal{I} \to A \to A/\mathcal{I} \to 0$. We shall show that $Ext^2_A(A/\mathcal{I},\, {}_AA^{\#}) = 0$. Indeed, by [11, Proposition 5.1.10], we obtain $Ext^2_A(A/\mathcal{I},\, {}_AA^{\#}) = Ext^1_A(\mathcal{I},\, {}_AA^{\#})$. But $Ext^1_A(\mathcal{I},\, {}_AA^{\#}) = 0$, by Corollary 4.8. Consequently, $inj.dim_A\,({}_AA^{\#}) \leqslant 1$. Since $inj.dim_A\,({}_AA^{\#}) \neq 0$, we obtain that $inj.dim_A\,({}_AA^{\#}) = 1$, as required. □

Take the quotient module $Q_1 = M_n(D)/{}_AA^{\#}$. We have an exact sequence $0 \to {}_AA^{\#} \to Q_0 = M_n(D) \to Q_1 \to 0$. By Proposition 4.17, we obtain that Q_1 is an injective A-module. Assume that the tiled order A is reduced. Then the injective hulls of simple A-modules $U_1, \dots, U_s$, by Lemma 4.14, may be written in the following form: $E(U_i) = e_{ii}M_n(D)/e_{ii}\Delta$, where e_{ii} are matrix idempotents, $i = 1, 2, \dots, n$.

5. Tiled orders and Frobenius rings

The finite Frobenius rings have many important applications in coding theory (see, for example, [5], [6], [8]).

In this section we shall construct following [7] a countable set of Frobenius quotient rings A_m with identity Nakayama permutation for any reduced tiled order A over a given discrete valuation ring $\mathcal{O}$. In particular, for any finite poset $\mathcal{P} = (p_1, \dots, p_n)$ we shall construct Frobenius rings $F_m(\mathcal{P})$ such that the quivers $Q(F_m(\mathcal{P}))$ of all rings $F_m(\mathcal{P})$ coincide. If $\mathcal{O}/\pi\mathcal{O}$ is a finite field then all Frobenius rings A_m are finite.

Denote by $\mathcal{P}_{max}$ the set of all maximal elements of $\mathcal{P}$, by $\mathcal{P}_{min}$ the set of all minimal elements of $\mathcal{P}$, and by $\mathcal{P}_{max} \times \mathcal{P}_{min}$ their Cartesian product.

To state the relationship between the quiver $Q(F_m(\mathcal{P}))$ and the poset $\mathcal{P}$ we recall the definition of the diagram of a poset $\mathcal{P}$.

The **diagram** of a poset $\mathcal{P} = (p_1, \dots, p_n)$ is the quiver $Q(\mathcal{P})$ with the set of vertices $VQ(\mathcal{P}) = \{1, \dots, n\}$ and the set of arrows $AQ(\mathcal{P})$ given as follows: there is an arrow from a vertex i to a vertex j if and only if $p_i \prec p_j$, and moreover, if $p_i \preceq p_k \preceq p_j$ then either $k = i$ or $k = j$.

The quiver $Q(F_m(\mathcal{P}))$ is obtained from the diagram $Q(\mathcal{P})$ by adding the arrows σ_{ij} for any $(p_i, p_j) \in \mathcal{P}_{max} \times \mathcal{P}_{min}$ (see [11, Theorem 14.6.3]).

Therefore, if $\mathcal{P}$ is a totally ordered set of n elements, then $Q(F_m(\mathcal{P}))$ is a simple cycle C_n, and hence all rings $F_m(\mathcal{P})$ are serial.

For any finite poset $\mathcal{P} = \{p_1, \dots, p_n\}$ we can construct a reduced tiled $(0,1)$-order $A(\mathcal{P})$ by setting

$$\mathcal{E}(A(\mathcal{P})) = (\alpha_{ij}),$$

where $\alpha_{ij} = 0 \Longleftrightarrow p_i \preceq p_j$ and $\alpha_{ij} = 1$, otherwise.

Then $A(\mathcal{P}) = \{\mathcal{O}, \mathcal{E}(A(\mathcal{P}))\}$ is a reduced $(0,1)$-order (see [11, §14.6]).

Theorem 5.1. *For any finite poset $\mathcal{P}$ there is a countable set of Frobenius rings $F_m(\mathcal{P})$ with identity Nakayama permutation such that $Q(F_m(\mathcal{P})) = Q(A(\mathcal{P}))$.*

Proof. Denote $A = A(\mathcal{P})$, $R = radA$, and $X = {}_AR^{\#}$. Let $\Delta = A_A^{\#}$ be the fractional ideal, as above. Then there exists a least positive integer t such that $\pi^t\Delta \subset R^2$. It is clear that $J = \pi^t\Delta$ is a two-sided ideal of $A(\mathcal{P})$. Write

$$F_m(\mathcal{P}) = A(\mathcal{P})/\pi^m J.$$

Since $\pi^m J \subset R^2$, it follows that $Q(F_m(\mathcal{P})) = Q(A(\mathcal{P}))$. The description of $Q(A(\mathcal{P}))$ is given by [11, Theorem 14.6.3]. The Artinian ring $F_m(\mathcal{P})$ is a Frobenius ring. Indeed, we have the following chain of inclusions:

$$A \supset R \supset R^2 \supset \pi^{m+t}X \supset \pi^m J.$$

Every indecomposable projective $F_m(\mathcal{P})$-module is of the form $\bar{P}_i = e_{ii}A/e_{ii}\pi^m J$. Therefore, $top\,\bar{P}_i = U_i$, and from Lemma 4.14 it follows that

$$soc\,\bar{P}_i = e_{ii}\pi^{m+t}X/e_{ii}\pi^{m+t}\Delta = U_i \quad \text{for } i = 1, \dots, n.$$

The same relation holds for the left modules. Therefore, the Nakayama permutation of $F_m(\mathcal{P})$ is identity. $\square$

Theorem 5.2. *For every reduced tiled order A over a discrete valuation ring, there is a countable set of Frobenius rings $F_m(A)$ with identity Nakayama permutation such that $Q(F_m(A)) = Q(A)$.*

Proof. For the fractional ideal Δ, there is the least positive integer t such that $\pi^t\Delta \subset R^2$. Then the quotient ring $Q(F_m(A)) = A/\pi^{m+t}\Delta$ is a Frobenius ring with the identity Nakayama permutation. $\square$

Example 1. Let k be a field, $\mathcal{O} = k[[x]]$ and $\pi = x$. Let

$$A = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \pi^\alpha\mathcal{O} & \mathcal{O} \end{pmatrix},$$

where $\alpha \geqslant 2$. Obviously,

$$\mathcal{E}(A) = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} \text{ and } [Q(A)] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

In this case

$$\mathcal{E}(\Delta) = \begin{pmatrix} 0 & -\alpha \\ 0 & 0 \end{pmatrix}.$$

We have

$$\mathcal{E}(R^2) = \begin{pmatrix} 2 & 1 \\ \alpha+1 & 2 \end{pmatrix}.$$

Consequently, $t = \alpha + 1$ and

$$\mathcal{E}(\pi^{\alpha+1}\Delta) = \begin{pmatrix} \alpha+1 & 1 \\ \alpha+1 & \alpha+1 \end{pmatrix},$$

and the quotient ring $F_m(A) = A/\pi^{m+t}\Delta$ is Frobenius with identity Nakayama permutation.

Note that

$$\mathcal{E}(\pi^{m+t}\Delta) = \begin{pmatrix} m+\alpha+1 & m+1 \\ m+\alpha+1 & m+\alpha+1 \end{pmatrix}.$$

Let k be a finite field with q elements. Then $F_m(A)$ is a finite Frobenius ring and $|F_m(A)| = q^{4m+3\alpha+4}$.

Theorem 5.3. *For any permutation $\sigma \in S_n$ there exists a countable set of Frobenius semidistributive algebras A_m such that $\nu(A_m) = \sigma$.*

Proof. Indeed, let $\mathcal{O}$ be a discrete valuation ring with the unique maximal ideal $\mathcal{M}$, and let

$$K_n(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathcal{M} & \dots & \mathcal{M} \\ \mathcal{M} & \mathcal{O} & \dots & \mathcal{M} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M} & \mathcal{M} & \dots & \mathcal{O} \end{pmatrix}$$

be a tiled order.

Let $\sigma : i \to \sigma(i)$ be a permutation of $\{1, \dots, n\}$ and let $\mathcal{I}_m = (\mathcal{M}^{w_{ij}})$ be a two-sided ideal of $K_n(\mathcal{O})$, where $w_{i\sigma(i)} = m + 1$, $w_{ij} = m$ for $j \neq \sigma(i)$ $(i, j = 1, \dots, n)$.

It is easy to see that $F_m(\mathcal{O}) = K_n(\mathcal{O})/\mathcal{I}_m$ is a Frobenius ring with Nakayama permutation σ. □

Let $\mathcal{O} = k[[t]]$ be a ring of formal power series over a field k, then $F_m(k[[t]]) = K_n(k[[t]])/I_m$ is a countable set of Frobenius semidistributive algebras $A_m = F_m(k[[t]])$ such that $\nu(A_m) = \sigma$. If k is finite, then all algebras A_m are finite.

Remark 5.4. Recall that QF-algebras with identity Nakayama permutation are called **weakly symmetric algebras**. Every weakly symmetric algebra is Frobenius. If for $\mathcal{O}$ we take the ring of formal power series $k[[t]]$ over a field k, then we obtain a countable series of weakly symmetric algebras A_m for every reduced tiled order over $k[[t]]$. If k is a finite field then all algebras A_m are finite.

6. Main Theorem

In this section we consider a special type of tiled orders which can be defined by the equivalent conditions of the following theorem:

Theorem 6.1. *The following conditions are equivalent for a tiled order A:*

(i) $inj.\,dim_A A_A = 1$;

(ii) $inj.\,dim_A\, {}_A A = 1$;

(iii) A^*_A *is projective left A-module;*

(iv) ${}_A A^*$ *is projective right A-module.*

Proof. (i)⇒ (iv). Denote $Q = Q_0 = M_n(D)$. By Proposition 4.1, Q is an injective right and left A-module. If $inj.\,dim_A\, A_A = 1$ then there exists an exact sequence

$$0 \to A_A \to Q_0 \to Q_0/A_A \to 0.$$

By [11, Proposition 6.5.5], the module Q_0/A_A is injective. Obviously, every indecomposable direct summand of Q_0/A_A has the form $e_{ii}Q_0/e_{ii}A$.

Since $e_{ii}Q_0/e_{ii}A$ is indecomposable injective then $soc\,(e_{ii}Q_0/e_{ii}A)$ is simple. Therefore every $e_{ii}A$ is a relatively injective irreducible A-lattice, by Proposition 4.16, and A_A is a relatively injective right A-module. By definition, $A_A \simeq_A P^*$. By duality properties ${}_AP =_A P_1 \oplus \ldots \oplus_A P_s \oplus P$, where ${}_AP_1, \ldots {}_A P_s$ are all pairwise non-isomorphic left principal A-modules, and every indecomposable direct summand of P is isomorphic to some ${}_AP_i$. Therefore, ${}_AA^*$ is a projective right A-module.

From Corollary 4.13 we have (iii)$\Leftrightarrow$ (iv). Finally, we obtain that (iv)$\Rightarrow$ (i), by Proposition 4.17 and the fact, that ${}_AA^*$ and A_A contain the same indecomposable summand if ${}_AA^*$ is projective. The case (ii)$\Leftrightarrow$ (iii) for left modules is proved as (i)$\Leftrightarrow$ (iv) for right modules. $\square$

Definition 6.2. *A tiled order A, which satisfies the equivalent conditions of Theorem 6.1, is called a* **Gorenstein tiled order***.*

As follows from Theorem 6.1 the definition of a Gorenstein tiled order is right-left symmetric.

Main Theorem *Let $A = \{\mathcal{O}, \mathcal{E}(A)\}$ be a reduced tiled order with the exponent matrix $\mathcal{E}(A) = (\alpha_{ij}) \in M_n(\mathbb{Z})$. A is Gorenstein if and only if the matrix $\mathcal{E}(A)$ is Gorenstein, i.e., there exists a permutation σ of the set $\{1, \ldots, n\}$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i, k = 1, \ldots, n$.*

Proof. Since A is reduced we have that $A^* \simeq A_A$. But

$$\mathcal{E}\left({}_AA\right) = \begin{pmatrix} 0 & \alpha_{12} & \ldots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \ldots & 0 \end{pmatrix}$$

and

$$\mathcal{E}\left({}_AA^*\right) = \begin{pmatrix} 0 & -\alpha_{21} & \ldots & -\alpha_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{1n} & -\alpha_{2n} & \ldots & 0 \end{pmatrix}.$$

Therefore, there exists a permutation σ of $\{1, \ldots, n\}$ such that

$$(\alpha_{i1}, \ldots, 0, \ldots, \alpha_{in}) = (-\alpha_{1\sigma(i)} + c_i, \ldots, -\alpha_{n\sigma(i)} + c_i),$$

where $c_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. Consequently, $\alpha_{ik} + \alpha_{k\sigma(i)} = c_i$ for $i, k = 1, \ldots, n$. For $i = k$ we obtain $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ and $\mathcal{E}(A)$ is a Gorenstein matrix. Conversely, if $\mathcal{E}(A)$ is Gorenstein then ${}_AA^* \simeq A_A$ and, by Theorem 6.1, the tiled order A is Gorenstein. $\square$

Example 2. Let $A = \sum_{i,j=1}^{n} \alpha_{ij} f_{ij} \subset M_n(Q)$, where $f_{ij} = c_{ij} e_{ij}$ for $i, j = 1, \dots, n$ and $\alpha_{ij} \in \mathbb{Z}$. Suppose that $c_{ij} = 1$ for $i \geqslant j$ and $c_{ij} = n$ for $i < j$. Then A is a $\mathbb{Z}$-order in $M_n(Q)$ of the following form.

$$A = \begin{pmatrix} \mathbb{Z} & n\mathbb{Z} & n\mathbb{Z} & \dots & n\mathbb{Z} & n\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & n\mathbb{Z} & \dots & n\mathbb{Z} & n\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \dots & n\mathbb{Z} & n\mathbb{Z} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \dots & \mathbb{Z} & n\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \dots & \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

with $n \in N$. Using Faddeev results on localizations [10, §2] and [12, Theorem 3.1] we obtain that $inj.\,dim._A A_A = 1$.

Note that the ring

$$\begin{pmatrix} \mathbb{Z} & 4\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

(see [3, Example 4.4]) and the ring

$$\begin{pmatrix} \mathbb{Z} & 4\mathbb{Z} & 4\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & 4\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

(see [4, Example 2.9]) are particular cases of our example.

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