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On the composite of three irreducible morphisms over string algebras

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Abstract. We characterize the representation-finite triangular string algebras having a path of irreducible morphisms of length three between pairwise non-isomorphic modules whose composite lies in the fourth power of the radical.

In the study of the module category over an algebra, the so-called irreducible morphisms and their composites play an important role. Recall that a morphism $f\colon X \longrightarrow Y$, where X and Y are modules over an algebra, is called *irreducible* provided it does not split and

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whenever $f = gh$, then either h is a split monomorphism or g is a split epimorphism. For indecomposable A-modules X, Y, denote by $\Re(X,Y)$ the set of all morphisms from X to Y which are not isomorphisms. Observe then that an irreducible morphism from X to Y belongs to $\Re(X,Y)$ but not to its square $\Re^2(X,Y)$. However, it is not true, in general, that a composite of $n \geq 2$ irreducible morphisms through indecomposable modules going from X to Y is either zero or lies in $\Re^n(X,Y) \setminus \Re^{n+1}(X,Y)$ (see examples in [6, 10]).

In the present paper we continue our study of composite of $n \geq 2$ irreducible morphisms started in [6, 7, 8]. In [6], we considered the situation of when the composite of two irreducible morphisms is non-zero and lies in $\Re^3(\mathrm{mod}A)$ (see also [1, 5]). In [7], we studied the more general situation of the composite of n irreducible morphisms lying in almost sectional paths, while in [8] we concentrate our attention for the particular case $n = 3$.

Continuing our work, here we will characterize the representation-finite triangular string algebras having a path of three irreducible morphisms whose composite is a non-zero morphism in $\Re^4$. String algebras were introduced by Butler-Ringel in [4] and the description of its modules given there will be very usufel for the problem we consider here. The analogous case of paths of irreducible morphisms of length two has been considered in [6].

1. Preliminaries

1.1. Throughout this paper, all algebras are finite dimensional basic k-algebras where k is a fixed algebraically closed field.

A *quiver* is given by two sets Q_0 and Q_1 together with two maps $s, e\colon Q_1 \longrightarrow Q_0$. The elements of Q_0 are called *points* while the elements of Q_1 are called *arrows*. For a given arrow $\alpha \in Q_1$, $s(\alpha)$ is the *starting point* of α and $e(\alpha)$ is the *ending point* of α. For each $\alpha \in Q_1$, denote by α^{-1} its formal inverse and define $s(\alpha^{-1}) = e(\alpha)$ and $e(\alpha^{-1}) = s(\alpha)$. A *walk* in Q is $c_1 \cdots c_n$, with $n \geq 1$, $e(c_i) = s(c_{i-1})$ for $1 \leq i \leq n$, and such that c_i is either an arrow or the inverse of an arrow. The inverse of a walk $c_1 \cdots c_n$ is $c_n^{-1} \cdots c_1^{-1}$. Finally, we say that $c_1 \cdots c_n$ is a *reduced walk* provided $c_{i+1} \neq c_i^{-1}$ for each i, $1 \leq i \leq n-1$. If A is an algebra, then there exists a quiver Q_A, called the *ordinary quiver of A*, such that A is the quotient of the path algebra kQ_A by an admissible ideal.

1.2. Let A be an algebra. We denote by mod$A$ the category of all finitely generated left A-modules, and by ind$A$ the full subcategory of mod$A$ consisting of one copy of each isomorphism class of the indecomposable A-module. Denote by Γ_A its Auslander-Reiten quiver, by τ the Auslander-Reiten translation DTr and by τ^{-1} its inverse. Let $X \in$ ind$A$. If X is not projective (not injective), denote by $\alpha(X)$ ($\alpha'(X)$, respectively) the number of indecomposable summands of the middle term of the almost split sequence ending (starting, respectively) at X.

A path $X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n$ of irreducible morphisms through indecomposable modules, with $n \geq 3$, is called *sectional* provided that for each $i = 3, \cdots, n$, X_{i-2} is not isomorphic to τX_i. By [9], the composite of irreducible morphisms lying in a sectional path of length n belongs to $\Re^n \setminus \Re^{n+1}$.

1.3. **String algebras.** A finite dimensional algebra $A = (kQ_A)/I_A$ is called *string* [4] provided:

(S1) Any point of Q_A is the starting point of at most two arrows.
(S1') Any point of Q_A is the ending point of at most two arrows.
(S2) Given an arrow β of Q_A, there is at most one arrow γ with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin I_A$.
(S2') Given an arrow β of Q_A, there is at most one arrow γ with $e(\beta) = s(\gamma)$ and $\gamma\beta \notin I_A$.
(S3) The ideal I_A is generated by a set of paths of Q_A.

1.4. **String modules.** Let $A = (kQ_A)/I_A$ be a string algebra. A *string* in Q_A is either a trivial path ϵ_v, $v \in Q_0$, or a reduced walk $C = c_n...c_1$ of length $n \geq 1$ such that no sub-walk $c_{i+t} \cdots c_{i+1}c_i$ nor its inverse belongs to I_A. We say that a string C *starts in a deep* (or *on a peak*) provided there is no arrow β such that $C\beta^{-1}$ ($C\beta$, respectively) is a string. Dually, a string C *ends in a deep* (or *on a peak*) provided there is no arrow β such that βC ($\beta^{-1}C$, respectively) is a string.

For each string $C = c_n...c_1$, one defines an indecomposable A-module $M(C)$ called *string.* We refer to [4] for details on this construction, as well for the results mentioned below.

The almost split sequences with indecomposable middle term and involving string modules are described as follows. Given $\beta \in (Q_A)_1$ such that there is a string $\gamma_r^{-1}...\gamma_1^{-1}\beta\delta_1^{-1}...\delta_s^{-1}$ which starts in a deep and ends on a peak (with $r, s \geq 0$ and γ_i, δ_j being arrows), then such a string is unique. We define the indecomposable string modules $N(\beta) = M(\gamma_r^{-1}...\gamma_1^{-1}\beta\delta_1^{-1}...\delta_s^{-1})$, $U(\beta) = M(\gamma_r^{-1}...\gamma_1^{-1})$ and $V(\beta) = M(\delta_1^{-1}...\delta_s^{-1})$. Then there exists an almost split sequence $0 \longrightarrow U(\beta) \longrightarrow N(\beta) \longrightarrow V(\beta) \longrightarrow 0$, and each almost split sequence involving string modules with indecomposable middle term is of this form.

For unexplained notions on representation theory, we refer to [2, 3, 11].

2. Composite of three irreducible morphisms

2.1. We first recall the following result from [8].

Theorem. *Let A be an algebra and let $X_1, X_2, X_3, X_4 \in indA$. Then there exist irreducible morphisms $h_i \colon X_i \longrightarrow X_{i+1}$, $i = 1, 2, 3$ such that $h_3h_2h_1 \neq 0$, $h_3h_2h_1 \in \Re^4(X_1, X_4)$, $h_2h_1 \notin \Re^3(X_1, X_3)$ and $h_3h_2 \notin \Re^3(X_2, X_4)$ if and only if one of the following conditions is satisfied.*

(a) *The path $X_1 \to X_2 \to X_3$ is sectional, $\alpha'(X_1) = 1$, $\alpha'(X_2) = 2$ and there are irreducible morphisms $f_i \colon X_i \to X_{i+1}$, $i = 1, 2, 3$ such that $f_3f_2f_1 = 0$ and $f_3f_2 \notin \Re^3(X_2, X_4)$, and a morphism $\varphi \in \Re^2(X_3, X_3)$ such that $f_3\varphi f_2f_1 \neq 0$; or*

(b) *The path $X_2 \to X_3 \to X_4$ is sectional, $\alpha(X_4) = 1$, $\alpha(X_3) = 2$ and there are irreducible morphisms $f_i \colon X_i \longrightarrow X_{i+1}$, $i = 1, 2, 3$ such that $f_3f_2f_1 = 0$ and $f_2f_1 \notin \Re^3(X_1, X_3)$, and a morphism $\varphi \in \Re^2(X_2, X_2)$ such that $f_3f_2\varphi f_1 \neq 0$.*

2.2. As a consequence, we can easily prove the following.

Proposition. *Let A be an algebra and $X_i \in indA$ for $i = 1, ..., 4$, with $X_2 \not\simeq X_4$. Then there exists a path $X_1 \xrightarrow{h_1} X_2 \xrightarrow{h_2} X_3 \xrightarrow{h_3} X_4$ of irreducible morphisms such that $h_3h_2h_1 \neq 0$, $h_3h_2h_1 \in \Re^4(X_1, X_4)$, $h_2h_1 \notin \Re^3(X_1, X_3)$ and $h_3h_2 \notin \Re^3(X_2, X_4)$ if and only if one of the following conditions is satisfied.*

(a) *The path $X_1 \to X_2 \to X_3$ is left almost sectional with $\alpha'(X_1) = 1$, $\alpha'(X_2) = 2$ and $\Re(X_1, X_4) \neq 0$; or*

(b) *The path* $X_2 \to X_3 \to X_4$ *is right almost sectional with* $\alpha(X_4) = 1$, $\alpha(X_3) = 2$ *and* $\Re(X_1, X_4) \neq 0$.

Proof. The necessity follows easily from (2.1).
For the sufficiency, assume that (a) holds. Note that X_i is not injective for $i = 1, 2$. Then, by [7] (Theorem 3.5), there exists a path $X_1 \xrightarrow{h_1} X_2 \xrightarrow{h_2} X_3 \xrightarrow{h_3} X_4$ of irreducible morphisms such that $0 \neq h_3h_2h_1 \in \Re^4$. It only remains to prove that $h_3h_2 \notin \Re^3$ and $h_2h_1 \notin \Re^3$. Clearly, $h_2h_1 \notin \Re^3$ because $X_1 \to X_2 \to X_3$ is a sectional path. Now, if $h_3h_2 \in \Re^3$, then by [10](Lemma 1.2), h_3 must be a surjective right minimal almost split morphism, a contradiction to the fact that $\alpha'(X_2) = 2$. A similar argument for (b) finishes the proof. $\square$

Observe that the cases (a) and (b) of the above result induce, respectively, the following configurations in Γ_A:

$$
\begin{tikzcd}
X_1 \arrow[dr] \arrow[rr, dotted, no head] & & \tau^{-1}X_1 \arrow[dr] & \\
& X_2 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & X_4 \\
& & X_3 \arrow[ur] &
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
& \tau X_4 \arrow[dr] \arrow[rr, dotted, no head] & & X_4 \\
X_1 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & X_3 \arrow[ur] & \\
& X_2 \arrow[ur] & &
\end{tikzcd}
$$

3. **The results**

3.1. In our first result, we give necessary and sufficient conditions for the existence of paths of irreducible morphisms between string modules of length three to have non-zero composite lying in $\Re^4$.

Proposition. *Let* $A = kQ_A/I_A$ *be a connected string algebra and* $X_i \in indA$, $i = 1, \cdots, 4$, *be pairwise non-isomorphic string modules. Then there exist irreducible morphisms* $h_i \colon X_i \longrightarrow X_{i+1}$, *with* $i = 1, 2, 3$ *such that* $h_3h_2h_1 \neq 0$, $h_3h_2h_1 \in \Re^4(X_1, X_4)$, $h_2h_1 \notin \Re^3(X_1, X_3)$ *and* $h_3h_2 \notin \Re^3(X_2, X_4)$ *if and only if there exist an arrow* $\beta_0 \in (Q_A)_1$, *a string* $C = \gamma_r^{-1} ... \gamma_1^{-1} \beta_0 \delta_1^{-1} ... \delta_s^{-1}$ *that starts in a deep and ends on a* *peak (with $r, s \geq 0$ and γ_i, δ_j being arrows) with $\gamma_1 ... \gamma_r \neq \delta_s ... \delta_1$ and satisfying one of the following conditions:*

(a) *C does not start on a peak, $M(C)$ is non-injective and there exists a string C' that starts in a deep with $C' = C\beta_1$ and $\beta_1 = \gamma_r$ or $C' = C\beta_1\nu_1^{-1}...\nu_t^{-1}$ (with β_1, ν_i being arrows and $t \geq 0$) such that either $e(\nu_t) = s(\gamma_r)$ or $\gamma_r^{-1}...\gamma_{r-l}^{-1} = \nu_{t-l}^{-1}...\nu_t^{-1}$ for some $l \in \mathbb{N}$, or if $t = 0$ then $e(\beta_1) = s(\gamma_r)$.*

(b) *C starts on a peak, $M(C)$ is non injective and $e(\delta_{s-1}) = s(\gamma_r)$ or $\gamma_r^{-1}...\gamma_{r-l}^{-1} = \delta_{s-1-l}^{-1}...\delta_{s-1}^{-1}$ for some $l \in \mathbb{N}_0$.*

(c) *C does not end in a deep, $M(C)$ is non projective and there exists a string C' that ends on a peak with $C' = \beta_1 C$ and $\beta_1 = \delta_s C$ or $C' = \nu_t^{-1}...\nu_1^{-1}\beta_1 C$ (with β_1, ν_i being arrows and $t \geq 0$) such that either $s(\delta_s) = e(\nu_t)$ or $\gamma_r^{-1}...\gamma_{r-l}^{-1} = \nu_{s-l}^{-1}...\nu_t^{-1}$ for some $l \in \mathbb{N}$ or if $t = 0$ then $s(\delta_s) = e(\beta_1)$.*

(d) *C ends in a deep, $M(C)$ is non projective and $e(\delta_s) = s(\gamma_{r-1})$ or $\gamma_{r-1}^{-1}...\gamma_{r-1-l}^{-1} = \delta_{s-l}^{-1}...\delta_s^{-1}$ for some $l \in \mathbb{N}_0$.*

Proof. Necessity. Observe that our assumption is equivalent to either (a) or (b) of (2.2). Assume we have condition (2.2)(a). Since X_2 is a string module and $\alpha(X_1) = 1$ then, as recalled above, there exists an arrow β_0 such that $X_2 = N(\beta_0) = M(\gamma_r^{-1}...\gamma_1^{-1}\beta_0\delta_1^{-1}...\delta_s^{-1})$ where $C := \gamma_r^{-1}...\gamma_1^{-1}\beta_0\delta_1^{-1}...\delta_s^{-1}$ is a string that starts in a deep and ends on a peak. Moreover, $X_1 = U(\beta_0) = M(\gamma_r^{-1}...\gamma_1^{-1})$ and $\tau^{-1}X_1 = V(\beta_0) = M(\delta_1^{-1}...\delta_s^{-1})$. Note that $M(C)$ is not injective and that $\tau^{-1}X_1 \not\simeq X_1$, since $\alpha(X_1) = 1$ and $X_4 \not\simeq X_2$. Then $\gamma_1...\gamma_r \neq \delta_s...\delta_1$.

Now, if C does not start on a peak and since C ends on a peak and $\alpha'(X_2) = 2$, by [4], the irreducible morphism $X_2 \to X_3$ is the canonical embedding from $M(C)$ to $M(C')$, where C' a string starting in a deep of either one of these forms $C' = C\beta_1$ or $C' = C\beta_1\nu_1^{-1}...\nu_t^{-1}$ (with β_1, ν_i being arrows and $t \geq 0$). Thus $X_4 = M(C'')$, with $C'' = \delta_1^{-1}...\delta_s^{-1}\beta_1$ if $C' = C\beta_1$ or $C'' = \delta_1^{-1}...\delta_s^{-1}\beta_1\nu_1^{-1}...\nu_t^{-1}$ if $C' = C\beta_1\nu_1^{-1}...\nu_t^{-1}$. By hypothesis, $\Re(X_1, X_4) \neq 0$ and $\beta_1 = \gamma_r$, in the first case. Otherwise $e(\nu_t) = s(\gamma_r)$ or $\gamma_r^{-1}...\gamma_{r-l}^{-1} = \nu_{t-l}^{-1}...\nu_t^{-1}$ for some $l \in \mathbb{N}$, or if $t = 0$, $e(\beta_1) = s(\gamma_r)$. Thus we get (a).

On the other hand, if C starts on a peak, and since C ends on a peak and $\alpha(X_2) = 2$, by [4] the irreducible morphism $X_2 \to X_3$ is the canonical projection $M(C) \to M(C')$ where $C' = \gamma_r^{-1}...\gamma_1^{-1}\beta_0\delta_1^{-1}...\delta_{s-1}^{-1}$ is a string not starting in a deep. Moreover,

$$0 \to M(C) \to \tau^{-1}X_1 \oplus M(C') \to M(\delta_1^{-1}...\delta_{s-1}^{-1}) \to 0$$

is the almost split sequence starting at X_2. Since $\Re(X_1, X_4) \neq 0$, then $\gamma_r^{-1}...\gamma_{r-l}^{-1} = \delta_{s-1-l}^{-1}...\delta_{s-1}^{-1}$ for some $l \in \mathbb{N}$ or $e(\delta_{s-1}) = s(\gamma_r)$. Thus we get (b).

If we now assume that condition (2.2)(b) holds then dual arguments lead to the cases (c) and (d).

Sufficiency. By hypothesis, since there is an arrow $\beta_0 \in (Q_A)_1$ such that $N(\beta_0) = M(\gamma_r^{-1}...\gamma_1^{-1}\beta_0\delta_1^{-1}...\delta_s^{-1})$ where $C = \gamma_r^{-1}...\gamma_1^{-1}$ $\beta_0\delta_1^{-1}...\delta_s^{-1}$ starts in a deep and ends on a peak, then there is an almost split sequence in mod$A$ with indecomposable middle term of the form

$$0 \to M(\gamma_r^{-1}...\gamma_1^{-1}) \xrightarrow{f} M(\gamma_r^{-1}...\gamma_1^{-1}\beta_0\delta_1^{-1}...\delta_s^{-1}) \xrightarrow{g} M(\delta_1^{-1}...\delta_s^{-1}) \to 0.$$

Suppose that (a) holds. By [4](lemma p. 166), since there are arrows $\beta_1, \nu_1, ..., \nu_t$ such that $C' = C\beta_1$ or $C' = C\beta_1\nu_1^{-1}\nu_t^{-1}$ are strings starting in a deep, then the canonical embedding $M(C) \to M(C')$ is irreducible. In particular, $M(C') \not\simeq M(\delta_1^{-1}...\delta_s^{-1})$ and $M(C)$ is not injective. Then $\alpha'(M(C)) = 2$ and there is a configuration of irreducible morphisms as in the condition (2.2)(a) as follows:

$$\begin{tikzcd}
M(\gamma_r^{-1}\cdots\gamma_1^{-1}) \arrow[dr, "f_1"] \arrow[rr, dotted, no head] & & M(\delta_1^{-1}\cdots\delta_s^{-1}) \arrow[dr, "f''"] & \\
& M(C) \arrow[ur, "f'"] \arrow[dr, "f_2"] \arrow[rr, dotted, no head] & & X_4 \\
& & M(C') \arrow[ur, "f_3"] &
\end{tikzcd}$$

with $X_4 \simeq M(\beta_1\delta_1^{-1}...\delta_s^{-1})$ or $X_4 \simeq M(\nu_t^{-1}...\nu_1^{-1}\beta_1\delta_1^{-1}...\delta_s^{-1})$ and $X_1 \simeq M(\gamma_r^{-1}...\gamma_1^{-1})$.

If $s(\nu_t) = e(\gamma_r)$ then $\mathrm{top}(M(\gamma_r^{-1}...\gamma_1^{-1})) = \mathrm{soc}(M(\nu_t^{-1}...\nu_1^{-1}\beta_1\delta_1^{-1}...\delta_s^{-1}))$. Therefore $\Re(X_1, X_4) \neq 0$.

Now, suppose that there is an $l \in \mathbb{N}$ such that $\gamma_r^{-1}...\gamma_{r-l}^{-1} = \nu_{t-l}^{-1}...\nu_t^{-1}$. Then there is an indecomposable module

$$N = M(\gamma_r^{-1}...\gamma_{r-l}^{-1}) = M(\nu_{t-l}^{-1}...\nu_t^{-1})$$

and non-isomorphisms $\varphi_1 = \pi f_2 f_1$ where

$$\pi : M(\gamma_r^{-1}...\gamma_1^{-1}\beta_0\delta_1^{-1}...\delta_s^{-1}) \to M(\gamma_r^{-1}...\gamma_{r-l}^{-1})$$

is the canonical projection to $M(\gamma_r^{-1}...\gamma_{r-l}^{-1})$ and $\varphi_2 = f_3\iota$ where

$$\iota : M(\nu_{t-l}^{-1}...\nu_t^{-1}) \to M(\gamma_r^{-1}...\gamma_1^{-1}\beta_1\nu_1^{-1}...\nu_t^{-1})$$

is the natural embedding. By [7](Theorem 2.4), since N is not isomorphic to $M(C')$ and $\varphi_1 \in \Re^3$, it is enough to show that $\varphi_2\varphi_1 \neq 0$ to get the result. If however $\varphi_2\varphi_1 = 0$ then $f_3\iota\pi f_2 f_1 = 0$ and hence

$$\mathrm{Im}(\iota\pi f_2 f_1) \subset \ \mathrm{Ker}\ f_3 \subset \ \mathrm{Ker} f'' \oplus \ \mathrm{Ker} f_3 = \ \mathrm{Im}\ (f', f_2)^t$$

and

$$\mathrm{Im}\ (f', f_2)^t = M(\delta_1^{-1}...\delta_s^{-1}) \oplus M(\gamma_r^{-1}...\gamma_1^{-1}\beta_0\delta_1^{-1}...\delta_s^{-1}) = X$$

But $\mathrm{Im}(\iota\pi f_2 f_1) = M(\gamma_r^{-1}...\gamma_{r-l}^{-1})$ is not a submodule of X and then $\varphi_2\varphi_1 \neq 0$, proving that $\Re(X_1, X_4) \neq 0$.

Now, suppose that condition (b) holds. Since, by hypothesis, C starts on a peak and $C' = \gamma_r^{-1}...\gamma_l^{-1}\beta_0\delta_1^{-1}...\delta_{s-1}^{-1}$ does not starts in a deep, then the canonical projection $M(C) \to M(C')$ is irreducible (see [4], lemma p. 169). Note that $M(C)$ is not injective, $\alpha'(M(C)) = 2$ and the irreducible morphisms are as in (2.2)(a). By hypothesis, $e(\delta_s) = s(\gamma_{r-1})$ or there is an $l \in \mathbb{N}$ such that $\gamma_r^{-1}...\gamma_{r-l}^{-1} = \delta_{s-l-1}^{-1}...\delta_{s-1}^{-1}$. In the former case, $\mathrm{top}X_1 = \mathrm{soc}X_4$. Therefore $\Re(X_1, X_4) \neq 0$. On the other hand, if there is an $l \in \mathbb{N}$ such that $\gamma_r^{-1}...\gamma_{r-l}^{-1} = \delta_{s-l-1}^{-1}...\delta_{s-1}^{-1}$, then there is an indecomposable module

$$N = M(\gamma_r^{-1}...\gamma_{r-l}^{-1}) = M(\delta_{s-l-1}^{-1}...\delta_{s-1}^{-1})$$

and non-isomorphisms $\varphi_1 = \pi f_2 f_1$ where

$$\pi : M(\gamma_r^{-1}...\gamma_1^{-1}\beta_0\delta_1^{-1}...\delta_{s-1}^{-1}) \to M(\gamma_r^{-1}...\gamma_{r-l}^{-1})$$

is the canonical projection to $M(\gamma_r^{-1}...\gamma_{r-l}^{-1})$ and $\varphi_2 = f_3\iota$ where

$$\iota : M(\delta_{s-l-l}^{-1}...\delta_{s-1}^{-1}) \to M(\gamma_r^{-1}...\gamma_1^{-1}\beta_0\delta_1^{-1}...\delta_{s-1}^{-1})$$

is the natural embedding. We claim that $0 \neq f_2 f_1$ is a monomorphism, since $\mathrm{Im} f_1 \bigcap \mathrm{Ker} f_2 = \{0\}$. It is enough to show that $\varphi_2\varphi_1 \neq 0$ to get the result. If however $\varphi_2\varphi_1 = 0$ then $f_3 \iota\pi f_2 f_1 = 0$ and hence

$$\mathrm{Im}(\iota\pi f_2 f_1) \subset \text{ Ker } f_3 \subset \mathrm{Ker} f'' \oplus \mathrm{Ker} f_3 = \text{ Im } (f', f_2)^t$$

and

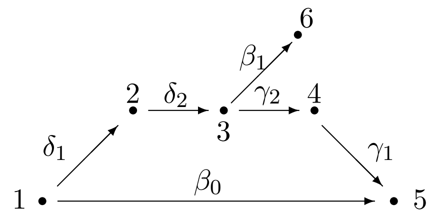
$$\begin{aligned} \text{Im } (f', f_2)^t &\simeq \tau^{-1} X_1 \oplus X_3 \\ &= M(\delta_1^{-1} ... \delta_s^{-1}) \oplus M(\gamma_r^{-1} ... \gamma_1^{-1} \beta_0 \delta_1^{-1} ... \delta_{s-1}^{-1}) \\ &= X. \end{aligned}$$

But $\mathrm{Im}(\iota\pi f_2 f_1) = M(\gamma_r^{-1} ... \gamma_{r-l}^{-1})$ is not a submodule of X and then $\varphi_2\varphi_1 \neq 0$, proving that $\Re(X_1, X_4) \neq 0$.

If we assume that conditions (c) or (d) hold, with a similar analysis as before we get the condition (2.2)(b). $\square$

3.2. The above result allows us to characterize the quivers of the representation - finite string algebras having paths of irreducible morphisms between non-isomorphic indecomposable modules of length three with non-zero composite lying in the fourth power of the radical. We will illustrate it in the following example.

Example. Let $A \simeq kQ_A/I_A$ be the k-algebra given by the quiver Q_A:



and I_A generated by $\gamma_2\delta_2 = 0$ and $\beta_1\delta_2 = 0$.

The string $C = \gamma_2^{-1}\gamma_1^{-1}\beta_0\delta_1^{-1}\delta_2^{-1}$ starts in a deep and ends on a peak, since there is no arrow β such that $C\beta^{-1}$ and $\beta^{-1}C$ are strings, respectively.

Observe that C does not end in a deep because there is an arrow β_1 such that $\beta_1 C = \beta_1\gamma_2^{-1}\gamma_1^{-1}\beta_0\delta_1^{-1}\delta_2^{-1}$ is a string. Moreover, $M(C)$

is non projective and $C' = \beta_1 C$ is a string that ends on a peak, with $\beta_1 \neq \delta_2$. Then by (c), since $e(\delta_2) = s(\beta_1)$ we conclude that there is a path of three irreducible morphisms with non-zero composite in $\Re^4$.

Before stating our main result, we will prove the following lemma.

Lemma. *Let $A \simeq kQ_A/I_A$ where Q_A is the bypass:*

$$\begin{tikzcd}
& 2 \arrow[r] & \bullet \cdots\cdots \bullet \arrow[r] & n \arrow[dr, "\alpha_n"] & \\
1 \arrow[ur, "\alpha_1"] \arrow[rrrr, "\beta"] & & & & n+1
\end{tikzcd}$$

with $n \geq 2$, and I_A is generated by a set of zero relations. Then, each pair of zero relations is overlapped in at least two arrows if and only if $J = \{i \in (Q_A)_0 : \alpha_n ... \alpha_i \notin I_A \ \text{ and } \alpha_{i-1} ... \alpha_1 \notin I_A\}$ is non-empty.

Proof. Assume that each pair of zero relations is overlapped in at least two arrows. Since Q_A is a bypass, then all the relations have to share at least the same two arrows. We may assume that such two arrows are α_j and α_{j-1}. Then clearly $\alpha_n \dots \alpha_j \notin I_A$ and $\alpha_{j-1} \dots \alpha_1 \notin I_A$. Therefore, $j \in J$ and it is not empty.

Now, if $J \neq \emptyset$, then there are a minimum $j \in J$ and a maximum $m \in J$. Then $\alpha_n \dots \alpha_m \dots \alpha_j \alpha_{j-1} \in I_A$ and $\alpha_m \dots \alpha_j \alpha_{j-1} \dots \alpha_1 \in I_A$. We claim that $\alpha_m \dots \alpha_{j-1}$ is a sub-path that belongs to each zero relation of I_A. In fact, assume that there is a zero relation γ which does not have such a sub-path. Since $\gamma \in I_A$, then if α_n is an arrow that belongs to γ we get a contradiction to the fact that $\alpha_n \dots \alpha_m \dots \alpha_j \notin I_A$. Otherwise, the relation $\alpha_n \dots \gamma \in I_A$. Now if $\gamma = \alpha_t \dots \alpha_{t-i}$ for $t - i \leq j$ we also get a contradiction with the fact that $\alpha_n \dots \alpha_m \dots \alpha_j \notin I_A$. In any other case we get a contradiction to the fact that $\alpha_{m-1} \dots \alpha_j \alpha_{j-1} \dots \alpha_1 \notin I_A$.

We observe that if $m = j$ then there are exactly two arrows in common in each zero relation. □

3.3. We now state and prove our main result.

Theorem. *Let $A = kQ_A/I_A$ be a connected triangular string algebra of finite representation type and $X_i \in indA$ pairwise non-isomorphic modules, for $i = 1, ..., 4$. Then there exist irreducible morphisms*

$h_i\colon X_i \longrightarrow X_{i+1}$ such that $h_3h_2h_1 \neq 0$, $h_3h_2h_1 \in \Re^4(X_1, X_4)$, $h_2h_1 \notin \Re^3(X_1, X_3)$ and $h_3h_2 \notin \Re^3(X_2, X_4)$ if and only if one of the following conditions holds:

(a) *The quiver Q_A has a bypass Q'_A :*

$$\begin{tikzcd}
& 2 \arrow[r] & \bullet \cdots\cdots \bullet \arrow[r] & n \arrow[dr, "\alpha_n"] & \\
1 \arrow[ur, "\alpha_1"] \arrow[rrrr, "\beta"] & & & & n+1
\end{tikzcd}$$

with $n \geq 2$, and such that $I_A \cap Q'_A$ is generated by zero relations satisfying that the set $J = \{i \in (Q'_A)_0 : \alpha_n...\alpha_i \notin I_A$ and $\alpha_{i-1}...\alpha_1 \notin I_A\}$ is non-empty.
If $J = \{a\}$ then one of the conditions (i) or (ii) is satisfied.
(i) there exist an arrow $\delta \neq \alpha_a$ with $s(\delta) = a$. If there is an arrow δ' such that $s(\delta') = e(\delta)$ then $\delta'\delta\alpha_{a-1}\cdots\alpha_1 = 0$ and there are no arrow δ'' with $e(\delta'') = e(\delta)$.
(ii) there exist an arrow $\mu \neq \alpha_{a-1}$ with $e(\mu) = a$. If there is an arrow δ' such that $e(\delta') = s(\mu)$ then $\alpha_n\cdots\alpha_a\mu\delta' = 0$ and there are no arrow δ'' with $s(\delta'') = s(\mu)$.

(b) *The quiver Q_A has a subquiver Q'_A :*

$$\begin{tikzcd}
1 \arrow[r, "\alpha_1"] \arrow[d, "\beta_0"'] & 2 \arrow[r] & \bullet \cdots \bullet \arrow[r, "\alpha_n"] & n+1 \\
n+r+2 & \bullet \arrow[l, "\nu_r"'] & \cdots\cdots \bullet & n+2 \arrow[l, "\nu_1"'] \arrow[u, "\beta_1"']
\end{tikzcd}$$

with $n \geq 2$, $r \geq 0$ and such that Q'_A is bounded by zero relations involving only arrows α_i, satisfying that $J = \{i \in (Q'_A)_0 : \alpha_n...\alpha_i \notin I_A$ and $\alpha_{i-1}...\alpha_1 \notin I_A\}$ is non-empty. In case that $J = \{a\}$, we have that $\delta\alpha_{a-1}\cdots\alpha_1 = 0$ and $\alpha_n\cdots\alpha_a\delta = 0$, for all arrows δ of Q_A.

(c) *The quiver Q_A has a subquiver Q'_A of the form:*

$$\begin{tikzcd}
1 \arrow[r, "\alpha_1"] \arrow[dd, "\beta_0"'] & \cdots \arrow[dr, "\alpha_{a-1}"'] & & \cdots \arrow[r, "\alpha_n"] & n+1 \\
& & a \arrow[ur, "\alpha_a"'] \arrow[dl, "\delta_a"] & & \\
n+r+2 & \cdots \arrow[l, "\delta_s"'] & & \cdots \arrow[ul, "\delta_{a-1}"] & n+2 \arrow[l, "\delta_1"'] \arrow[uu, "\beta_1"']
\end{tikzcd}$$

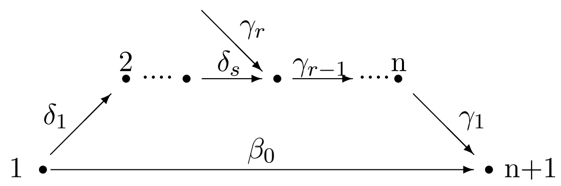
such that $I_A \cap Q'_A$ is generated by zero relations, the set $J = \{i \in (Q'_A)_0 : \alpha_n ... \alpha_i \notin I_A \quad and\ \alpha_{i-1}...\alpha_1 \notin I_A\}$ is non-empty, $\delta_a \alpha_{a-1} = 0$, $\alpha_a \delta_{a-1} = 0$ and $\delta_s ... \delta_1 \notin I_A$. In case that $J = \{b\}$, with $a \neq b$, then $\delta \alpha_{b-1} \cdots \alpha_1 = 0$ and $\alpha_n \cdots \alpha_b \delta = 0$, for all arrows δ of Q_A.

Proof. Necessity. By (3.1), there exists an arrow $\beta_0 \in (Q_A)_1$ and a string $C = \gamma_r^{-1}...\gamma_1^{-1}\beta_0\delta_1^{-1}...\delta_s^{-1}$ that starts in a deep and ends on a peak with $\gamma_1...\gamma_r \neq \delta_s...\delta_1$.

Assume first that condition (a) of (3.1) holds. Then C does not start on a peak, and there is an arrow β_1 such that $C\beta_1$ is a string. If $\beta_1 = \gamma_r$ then $C' = C\beta_1$ is a string that starts in a deep, and so we infer that there is a walk Q'_A in Q_A as follows

$$\xrightarrow{\beta_1} . \xleftarrow{\delta_s} \ldots \xleftarrow{\delta_1} . \xrightarrow{\beta_0} . \xleftarrow{\gamma_1} \ldots \xleftarrow{\gamma_r}$$

Moreover, since A is triangular and C does not start on a peak, it implies that Q'_A is indeed a subquiver as follows



where $kQ'_A \bigcap I_A =< \gamma >$ with $\ell(\gamma) = 2$, because A is a string algebra and $\gamma_{r-1}\gamma_r \neq 0$.

On the other hand, since C ends on a peak, then C' ends on a peak and if there is an arrow δ such that $e(\delta) = s(\gamma_r)$ then $\gamma_r \delta \in I_A$. Observe that since $C' = C\gamma_r$ is a string that starts in a deep then there is no arrow δ such that $s(\delta) = s(\gamma_r)$. Thus, (a) is satisfied.

Now, if $\beta_1 \neq \gamma_r$ then C' is of the form

$$C' = \gamma_r^{-1}...\gamma_1^{-1}\beta_0\delta_1^{-1}...\delta_s^{-1}\beta_1\nu_1^{-1}...\nu_t^{-1}$$

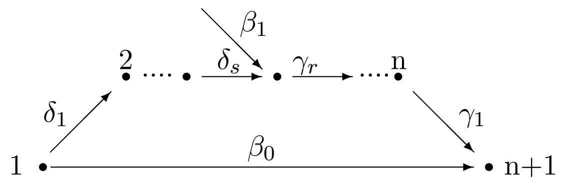
and we infer that there is a walk Q'_A in Q_A as follows

$$\xleftarrow{\nu_t} \ldots \xleftarrow{\nu_1} . \xrightarrow{\beta_1} . \xleftarrow{\delta_s} \ldots \xleftarrow{\delta_1} . \xrightarrow{\beta_0} . \xleftarrow{\gamma_1} \ldots \xleftarrow{\gamma_r}$$

with $t \geq 0$. If $t = 0$ and C' is a string with the condition $e(\beta_1) = s(\gamma_r)$, then we infer that there is a walk Q'_A in Q_A as follows

$$\xrightarrow{\beta_1} . \xleftarrow{\delta_s} ... \xleftarrow{\delta_1} . \xrightarrow{\beta_0} . \xleftarrow{\gamma_1} ... \xleftarrow{\gamma_r} .$$

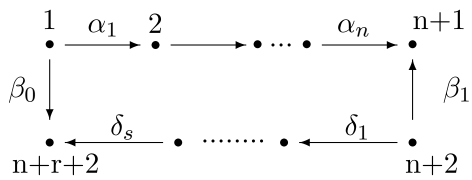
Then it implies that Q'_A is a sub-quiver of the form:



where $kQ'_A \bigcap I_A =< \gamma >$ with $\ell(\gamma) = 2$, since C is a string that ends on a peak. Hence $J = \{a\}$ and $\gamma_r\delta_s = 0$. Moreover, because C ends on a peak then we also have that $\gamma_r\beta_1 = 0$. If there is an arrow δ' such that $e(\delta') = s(\beta_1)$ then $\delta'\beta_1\gamma_r \cdots \gamma_1 = 0$ since $'\beta_1\gamma_r = 0$ and there are no arrow δ'' with $s(\delta'') = s(\beta_1)$, since C' starts in a deep.

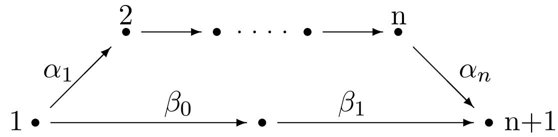
Now, if $t > 0$ then by hypothesis, since A is triangular, C and C' are strings with the given conditions, $\gamma_r^{-1}...\gamma_{r-l}^{-1} = \nu_{t-l}^{-1}...\nu_t^{-1}$ or $e(\nu_t) = s(\gamma_r)$, then it implies that Q'_A is indeed one of the following sub-quivers (1),(2),(3),(4) or (5) below (identifying vertices in the cases that it is possible) and denoting the arrows $\nu_1, ...\nu_t, \gamma_1, ..., \gamma_t$ by $\alpha_1, ..., \alpha_n$:

(1)



Observe that $\delta_s...\delta_1 \notin I_A$. In particular when $\delta_s...\delta_1$ is a point then $\beta_1\beta_0 \neq 0$ since C' is a string and (1) is the subquiver

(2)



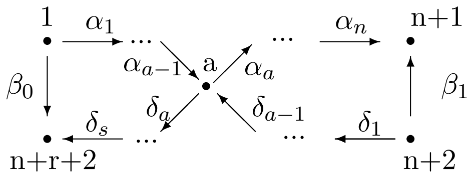
In both cases, since C' starts in a deep and ends on a peak then $kQ'_A \bigcap I_A$ is non-zero and the set

$$J = \{i \in (Q'_A)_0 : \alpha_n ... \alpha_i \notin I_A \ \ and \ \ \alpha_{i-1} ... \alpha_1 \notin I_A\}$$

is clearly non-empty. Observe that since $\delta_s ... \delta_1 \notin I_A$ then $\alpha_n ... \alpha_1 = 0$. Suppose that $kQ'_A \bigcap I_A =< \gamma >$ where $\ell(\gamma) = 2$, then $J = \{a\}$. If $\delta \in (Q'_A)_1$ with $s(\delta) = a$ and $\delta \neq \alpha_a$ then $\delta\alpha_{a-1} \cdots \alpha_1 \in I_A$, because C' starts in a deep. Dually, if $\delta \in (Q'_A)_1$ with $e(\delta) = a$ and $\delta \neq \alpha_{a-1}$ then $\alpha_1 \cdots \alpha_a \delta \in I_A$ (observe that C' ends on a peak because C ends on a peak). Therefore we get condition (b).

We can also identified some vertices in (1) getting the following subquiver in Q_A:

(3)



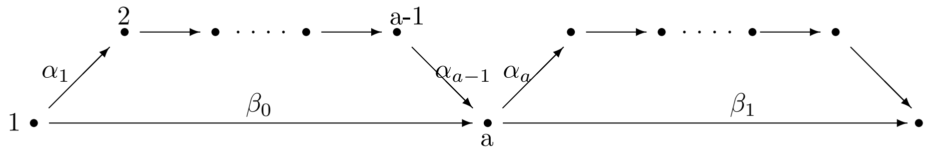
Since C' starts in a deep and ends on a peak and $\delta_s \ldots \delta_1 \notin I_A$ then $kQ'_A \bigcap I_A$ is non-zero and the set

$$J = \{i \in (Q'_A)_0 : \alpha_n ... \alpha_i \notin I_A \ \ and \ \ \alpha_{i-1} ... \alpha_1 \notin I_A\}$$

is non-empty. Moreover, since A is a representation-finite string algebra, then $\delta_a \alpha_{a-1} = 0$ and $\alpha_a \delta_{a-1} = 0$. Doing a similar analysis as the one made for the subquivers (1) and (2) when $J = \{a\}$ we get (c).

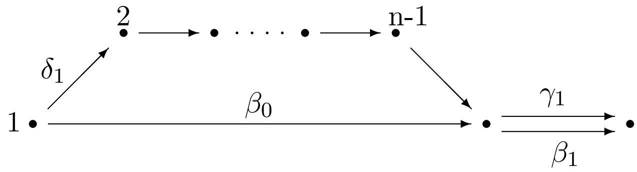
Besides the quivers above, we have the following possibilities

(4)



and

(5)



which gives, clearly, representation-infinite algebras, a contradiction to our hypothesis.

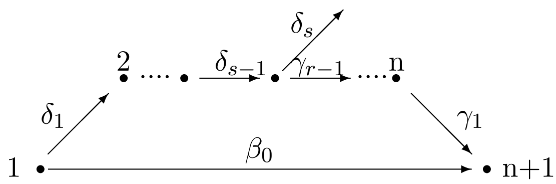
Now, assume that condition (b) of (3.1) holds. Then

$$C = \gamma_r^{-1} \ldots \gamma_1^{-1} \beta_0 \delta_1^{-1} \ldots \delta_s^{-1}$$

is a string that starts on a peak, and we infer that there is a walk Q'_A in Q_A of the form:

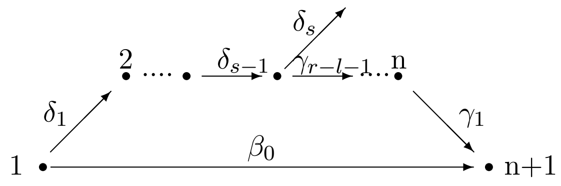
$$\xleftarrow{\delta_s} \ldots \xleftarrow{\delta_1} . \xrightarrow{\beta_0} . \xleftarrow{\gamma_1} \ldots \xleftarrow{\gamma_r}$$

First, if $e(\delta_{s-1}) = s(\gamma_r)$ and since A is triangular then we get a subquiver as follows

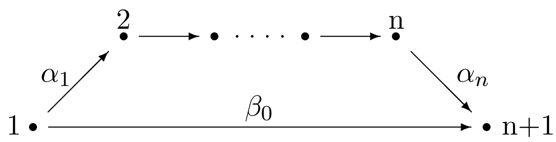


where $\delta_s ... \delta_1 \notin I_A$ and $kQ'_A \bigcap I_A =< \gamma >$ with $\ell(\gamma) = 2$, since C ends on a peak and A is a string algebra. Moreover, by the conditions of the strings C and C', we have that the conditions of (a)(i) hold.

If $\gamma_r^{-1} ... \gamma_{r-l}^{-1} = \delta_{s-1-l}^{-1} ... \delta_{s-1}^{-1}$ for some $l \geq 0$ then $\gamma_{r-l-1} = \delta_s$, since otherwise we have a subquiver of the form:



with $\delta_s \delta_{s-1} \in I_A$ since A is a string algebra, a contradiction to the fact that C is a string. Moreover, we proved that if $\gamma_r^{-1} ... \gamma_{r-l}^{-1} = \delta_{s-1-l}^{-1} ... \delta_{s-1}^{-1}$ then $\gamma_r^{-1} ... \gamma_{r-l}^{-1} \gamma_{r-l-1}^{-1} = \delta_{s-1-l}^{-1} ... \delta_{s-1}^{-1} \delta_s^{-1}$, for some $l \geq 0$. Then Q'_A is indeed a bypass as follows



with $\ell(\gamma) > 2$. Denote by $a = \min J$ and $b = \max J$. Since C starts on a peak, there are no arrow ending in b, but anyway we can also have three irreducible morphisms with non zero composite in $\Re^4$ when we consider the string given in (3.1)(d). In this case, we find that such irreducible morphisms are in a configuration as described in (2.2)(b), proving the implication.

Now, if (3.1)(c) holds, since C does not ends on a peak, then there is an arrow β_1 such that $\beta_1^{-1} C$ is a string. In a similar way as before we can prove that conditions (c) or (d) of (3.1) imply (2.2)(a) and (2.2)(b), which finishes this implication.

Sufficiency. To prove this implication, we are going to show that for each subquiver stated, we can find strings as in Proposition 3.1. Since we get the subquivers using the fact that there has to be a particular string on it, then clearly we can recover it from the quiver proving all the cases. We are going to prove here only one case to show the technique and we left the other cases to the reader.

Assume that (a) holds. First, consider the case that $b = \max J$, $a = \min J$ and $a \neq b$. Then there is an $l > 0$ such that $\gamma_{r-l}...\gamma_r = \delta_{s-1}...\delta_{s-1-l}$ and we have a string:

$$C : b \overset{\delta_s}{\leftarrow} ... \overset{\delta_1}{\leftarrow} . \overset{\beta_0}{\rightarrow} . \overset{\gamma_1}{\leftarrow} ... \overset{\gamma_r}{\leftarrow} a$$

that starts in a deep and ends on a peak. Moreover, C is a string starting on a peak, and (3.1)(b) holds. Now, if $\max J = \min J = a$, then the string

$$C : b \overset{\delta_s}{\leftarrow} ... \overset{\delta_1}{\leftarrow} . \overset{\beta_0}{\rightarrow} . \overset{\gamma_1}{\leftarrow} ... \overset{\gamma_r}{\leftarrow} a$$

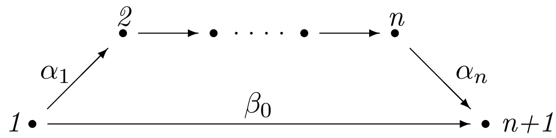
starts in a deep and ends on a peak. Moreover, by hypothesis if (i) holds then C is a string not starting on a peak. Then there is an arrow β_1 and a string $C' = C\beta_1$ or $C' = C\beta_1\nu_1^{-1}...\nu_t^{-1}$ that starts in a deep. Clearly, $M(C)$ is not injective and we get condition (3.1)(a).

In a similar way, we can analyzed the rest of the cases getting the result. □

3.4. In [6], we gave a characterization of the quivers of triangular string algebras of finite representation type having two irreducible morphisms between indecomposable modules with non zero composite in $\Re^3$. Using this result and the above theorem we can obtain a characterization of the quivers of triangular string algebras of finite representation type having three irreducible morphisms between non-isomorphic indecomposable modules with non zero composite in $\Re^4$, but not two with non-zero composite in $\Re^3$.

Corollary. *Let $A = kQ_A/I_A$ be a connected triangular string algebra of finite representation type and $X_i \in indA$ pairwise non-isomorphic modules, for $i = 1, ..., 4$. Then there are irreducible morphisms $h_i : X_i \to X_{i+1}$ for $i = 1, 2, 3$ with non-zero composite in $\Re^4(X_1, X_4)$ and for any two irreducible morphisms we have that their composite is not in $\Re^3$ if and only if the quiver Q_A does not have a bypass as in (3.2)(d) and one of the following holds:*

(a) *The quiver Q_A has a bypass Q'_A :*

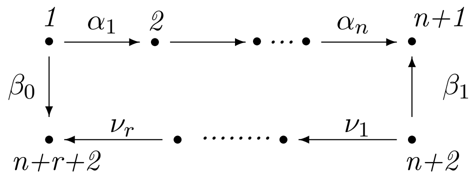


where $I_A \cap Q'_A$ is generated by zero relations such that the set $J = \{i \in (Q'_A)_0 : \alpha_n ... \alpha_i \notin I_A$ and $\alpha_{i-1}...\alpha_1 \notin I_A\} = \{a\}$ and satisfying one of the following conditions:

(i) there exist an arrow $\delta \neq \alpha_a$ with $s(\delta) = a$. If there is an arrow δ' such that $s(\delta') = e(\delta)$ then $\delta'\delta = 0$ and there are no arrow δ'' with $e(\delta'') = e(\delta)$.

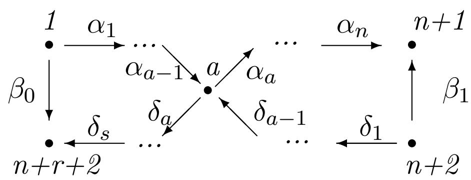
(ii) there exist an arrow $\mu \neq \alpha_{a-1}$ with $e(\mu) = a$. If there is an arrow δ' such that $e(\delta') = s(\mu)$ then $\mu\delta' = 0$ and there are no arrow δ'' with $s(\delta'') = s(\mu)$.

(b) *The quiver Q_A has a sub-quiver Q'_A :*



with $n \geq 2$ and $r \geq 0$ bounded by zero relations involving only arrows α_i satisfying that the set $J = \{i \in (Q'_A)_0 : \alpha_n...\alpha_i \notin I_A$ and $\alpha_{i-1}...\alpha_1 \notin I_A\}$ is non-empty. In case that $J = \{a\}$, then $\delta\alpha_{a-1}\cdots\alpha_1 = 0$ and $\alpha_1\cdots\alpha_a\delta = 0$, for all arrows δ in Q_A.

(c) *The quiver Q_A has a subquiver Q'_A :*



with Q'_A bounded by zero relations, satisfying that $J = \{i \in (Q'_A)_0 : \alpha_n...\alpha_i \notin I_A$ and $\alpha_{i-1}...\alpha_1 \notin I_A\}$ is non-empty,

$\delta_a\alpha_{a-1} = 0$, $\alpha_a\delta_{a-1} = 0$ and $\delta_s ... \delta_1 \notin I_A$. In case that $J = \{b\}$, with $b \neq a$, then $\delta\alpha_{b-1}\cdots\alpha_1 = 0$ and $\alpha_n\cdots\alpha_b\delta = 0$, for all arrows δ of Q_A.

Proof. The result follows easily from (3.3) and [6](Proposition 3.4). □

3.5. We finish this paper with some examples.

Examples. (a) Let A be the algebra given by the quiver:

$$\begin{tikzcd}
& \bullet\, 2 \arrow[dr, "\beta"] & \\
1\,\bullet \arrow[ur, "\alpha"] \arrow[r] & \overset{3}{\bullet} \arrow[r] & \bullet\, 4
\end{tikzcd} \qquad \text{with } \beta\alpha = 0$$

For this algebra, there are paths of three irreducible morphisms whose composition is non-zero but lying in the fourth power of the radical. On the other hand, there are no pairs of irreducible morphisms whose composite is non-zero and lies in $\Re^3$. We refer to [8] for details on the latter remark.

(b) Let now A be the k-algebra given by the quiver:

$$\begin{tikzcd}
& \overset{2}{\bullet} \arrow[r, "\alpha_2"] & \overset{3}{\bullet} \arrow[r, "\alpha_3"] & \overset{4}{\bullet} \arrow[r, "\alpha_4"] & \overset{5}{\bullet} \arrow[dr, "\alpha_5"] & \\
1\,\bullet \arrow[ur, "\alpha_1"] \arrow[rrrrr, "\beta"] & & & & & \bullet\, 6
\end{tikzcd}$$

with $\alpha_5\alpha_4\alpha_3\alpha_2 = 0$ and $\alpha_4\alpha_3\alpha_2\alpha_1 = 0$. Denoting by P_j (or I_j) the indecomposable projective (injective, respectively) associated with the vertex j, then there are almost split sequences

$$0 \longrightarrow P_3 \xrightarrow{f_1} M_1 \xrightarrow{h_1} I_4 \longrightarrow 0 \text{ and}$$

$$0 \longrightarrow M_1 \xrightarrow{(h_1;f_2)^t} I_4 \oplus M_2 \xrightarrow{(h_2,f_3)} I_3 \longrightarrow 0$$

where $M_1, M_2 \in \mathrm{ind}A$. Observe that $\Re(P_3, I_4) \neq 0$, and so, by [6](2.4) there are irreducible morphisms $P_3 \xrightarrow{g_1} M_1 \xrightarrow{g_2} I_4$ with a non-zero composite in $\Re^3$. Also, since $\Re(M_2, M_2)$ has a non-zero morphism φ, one can easily see (as in the above example) that the composite of the irreducible morphisms f_3, $f_2 + \varphi f_2$ and f_1 is non-zero and lies in $\Re^4$.

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