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Existence of solutions for singular fully nonlinear equations

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Abstract. In this note we describe how to approximate some classes of singular equations by nonsingular equations. We obtain a solution to each nonsingular problem and estimates guaranteeing that the limiting function is a solution of the original problem.

1. Introduction

The following problem was studied in [5]

$$\begin{cases} -\Delta u = \chi_{\{u>0\}}\big(-u^{-\beta}+\lambda u^p\big) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

$0<\beta<1$ and $0<p<1$.

Theorem 1.1. *There exists a maximal solution for every $\lambda>0$. There is constant $\lambda^*>0$ such that for $\lambda>\lambda^*$ the maximal solution is positive. And for $\lambda<\lambda^*$, the maximal solution vanishes on a set of positive measure.*

We solve problem (1) by perturbing the equation as $-\Delta u + \frac{u}{(u+\varepsilon)^{1+\beta}} = \lambda u^p$. The solutions $u_\varepsilon \searrow u$ pointwise and

$$\int_\Omega u(-\Delta\varphi) + \int_{\{u>0\}} \frac{1}{u^\beta}\varphi \le \lambda \int_\Omega u^p \varphi, \tag{2}$$

$\forall \varphi \in C^2(\overline{\Omega})$, $\varphi \ge 0$, $\varphi = 0$ on $\partial\Omega$.

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There are two approaches to show that u is indeed a solution of (1). Relation (2) tells us that u is a maximal subsolution. We then regularize it and show that $u \in C^{1,\frac{1-\beta}{1+\beta}}$ and indeed solves the problem (1). In doing this, we need to obtain a local estimate $|\nabla u| \leq Cu^{\frac{1-\beta}{2}}$ in $\Omega' \subset\subset \Omega$. One of the main ingredients is the following Harnack type lemma.

Lemma 1.2. *For every ball $B_r(p) \subset \Omega$ there are constants $c_0, \tau > 0$ depending only on n and β such that if*

$$\fint_{\partial B_r(p)} u \geq c_0 r^{\frac{2}{1+\beta}}, \ \textit{then } u(x) \geq \tau \fint_{\partial B_r(p)} u \quad \textit{a.e. in } B_{r/2}(p)$$

The second approach relies on an estimate for u_ε by the maximum principle, namely $|\nabla u_\varepsilon| \leq C u_\varepsilon^{\frac{1-\beta}{2}}$ in $\Omega' \subset\subset \Omega$. The idea to obtain such an estimate is to define $v = \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^{1-\beta}} \varphi_1^2$, where φ_1 is the first eigenfunction of the Laplacian with zero boundary condition. The function v has a maximum at $x_0 \in \Omega$, and then $\Delta v(x_0) \leq 0$. If the estimate is not true, it is possible to take a constant $C > 0$ independently of ε such that $\sup v > C$ and by computation $\Delta v(x_0) > 0$, a contradiction. Using the estimate and multiplying the equation by an adequate test function, we let $\varepsilon \to 0$ in the equation to get a weak solution.

The next problem was studied in [7]

$$\begin{cases} -\Delta u = \chi_{\{u>0\}} \big(\log u + \lambda u^p \big) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

Both approaches described above work in this case and an analogous result to Theorem 1.1 holds true. The estimate obtained for the maximal subsolution (which is shown to be a solution) is $|\nabla u| \leq Cu$ in $\Omega' \subset\subset \Omega$ and $u \in C^{1,1}$, a better regularity than the one for (1). This is roughly explained by the fact that *log u* is less singular than $-1/u^\beta$. The estimate by maximum principle is $|\nabla u_\varepsilon| \leq C u_\varepsilon$ in $\Omega' \subset\subset \Omega$.

2. Fully nonlinear elliptic equations

We proceed to discuss in more detail the following fully nonlinear problem which was addressed in a work in progress with E. Teixeira [8]. We consider

$$\begin{cases} F(D^2 u) &= & G\big(x, u, |\nabla u|^2\big) & \text{in} & \Omega \\ \qquad u &= & f & \text{on} & \partial\Omega \end{cases}$$

with $f \in C^{1,\alpha}(\partial\Omega)$ and $G\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a C^1 function. Following [3], we define $F\colon \mathrm{Sym}(d \times d) \to \mathbb{R}$ and assume $F(0) = 0$. The uniform ellipticity reads as follows: $\exists\lambda, \Lambda$, $0 < \lambda \leq \Lambda$ such that

$$F(\mathcal{M}+\mathcal{N}) \leq F(\mathcal{M}) + \Lambda\|\mathcal{N}^+\| - \lambda\|\mathcal{N}^-\|, \ \forall \mathcal{M}, \mathcal{N} \in \mathrm{Sym}(d \times d).$$

In order to state our Lipschitz estimate, let $\phi\colon (0,\infty) \to \mathbb{R}$ be such that $\liminf\limits_{s\to\infty} \phi(s) \geq 0$. We define the asymptotic behavior of ϕ passing 0, $\kappa\colon (0,1) \to (0,\infty)$ by $\kappa(\varepsilon) := \inf\{s : \phi(s) > -\varepsilon\}$.

Theorem 2.1. *Let $u \in C^3(\Omega)$ be a solution. Define*

$$\sigma(|p|) := \inf_{(x,u)} \frac{D_u G(x,u,|p|^2)|p|^2 - \left|D_x G(x,u,|p|^2)\right| |p|}{G^2(x,u,|p|^2)}$$

assume $S := \liminf\limits_{|p|\to\infty} \sigma(|p|) \geq 0$. Then $\max\limits_{\overline{\Omega}} |\nabla u| \leq C$, where C depends only on d, λ, Λ, $\|f\|_{C^{1,\alpha}}$ and the asymptotic behavior of σ passing 0.

The proof runs by defining $v = |\nabla u|^2$. We compute $D_{i,j}v$ and use the equation. Since v has a maximum at $x_0 \in \Omega$, we use the asymptotic behavior to conclude the estimate. It is not a proof by contradiction.

Specializing the function G we study the problem

$$\left\{ \begin{array}{rcll} F(D^2u) & = & \beta(u)\Gamma\left(|\nabla u|^2\right) & \text{in} \quad \Omega \\ u & = & f & \text{on} \quad \partial\Omega, \end{array} \right. \tag{3}$$

where $\beta\colon \mathbb{R} \to \mathbb{R}$ and $\Gamma\colon [0,\infty) \to \mathbb{R}$ are $C^{1,\alpha}$ functions. We have two consequences of Theorem 2.1.

Corollary 2.2. *If $\inf_u \frac{\beta'(u)}{\beta(u)^2} > -\infty$ and $\frac{\Gamma(\tau)}{\tau} \to +\infty$ as $\tau \to +\infty$, then $\max\limits_{\overline{\Omega}} |\nabla u| \leq C$.*

Corollary 2.3. *If β is nondecreasing, $|\beta| + |\beta'| > 0$ and $\liminf\limits_{\tau\to\infty} \Gamma(\tau) > 0$, then $\max\limits_{\overline{\Omega}} |\nabla u| \leq C$.*

Definition: u is a viscosity subsolution in Ω if $F(D^2u) \geq g$ in the viscosity sense in Ω if, that is, for every $x_0 \in \Omega$, V_{x_0} neighborhood, $\varphi \in C^2(V_{x_0})$, $u \leq \varphi$ in V_{x_0}, $u(x_0) = \varphi(x_0)$, then $F(D^2\varphi(x_0)) \geq g(x_0)$.

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A viscosity solution is a continuous function u which is a subsolution and a supersolution.

Definition: a continuous function u satisfies $F(D^2u) = +\infty$ at a point X_0 if u cannot be touched from above by a smooth function at X_0.

The idea behind the above definition relies in the fact that if $u(x) = |x|$, then $\Delta u = (n-1)/|x|$ in the distributional sense. We could say that $\Delta u(0) = +\infty$. In the light of the viscosity theory, given an arbitrary positive number K, $P_K(x) = \frac{K}{2n}|X|^2$ touches u at 0 from below. Indeed, $P(0) = u(0)$ and in $0 < |x| < \frac{1}{K}$, we have $u(x) > P_K(x)$. Thus, "$\Delta u(0) \geq K$" for every K.

Definition: a continuous function u is a viscosity solution in the topological sense if it satisfies $F(D^2u) = +\infty$ at a point X_0.

The approach to solve (3) is by considering again a perturbed problem

$$\begin{cases} F\left(D^2 u_\epsilon\right) &= \beta_\epsilon(u_\epsilon)\Gamma\left(|\nabla u_\epsilon|^2\right) \quad \text{in} \quad \Omega \\ u_\epsilon &= f \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4}$$

Using Corollary 2.2 we derive existence of a Lipschitz viscosity solution in the topological sense for

$$\begin{cases} F(D^2u) &= \dfrac{1}{|u|^q}\Gamma\left(|\nabla u|^2\right) \quad \text{in} \quad \Omega \\ u &= f \quad \text{on} \quad \partial\Omega, \end{cases} \tag{5}$$

with $q \geq 1$, $\Gamma \geq 0$, Γ superlinear and F concave. In this case $\beta_\epsilon(u) = 1/u^q$ for $u > \varepsilon$ and $\beta_\epsilon(u) = \varepsilon$ for $u < -\varepsilon$. Between $-\varepsilon$ and ε, $\beta_\epsilon(u)$ is a fourth order polinomial. Since $\beta_\epsilon(u)$ is not monotone, Perron's method should be adapted by adding a term ku in both sides of the equation. This gives a solution u_ε to (4). The estimate of Theorem 2.1 allows us to let $u_\varepsilon \to u$, thus obtaining a viscosity solution of (5).

Another existence of viscosity solution result can be obtained using Corollary 2.3 for the problem

$$\begin{cases} F(D^2u) &= \chi_{\{u>0\}}\Gamma\left(|\nabla u|^2\right) \quad \text{in} \quad \Omega \\ u &= f \quad \text{on} \quad \partial\Omega. \end{cases} \tag{6}$$

In this case β_ε is defined as follows. Let ρ be a smooth function supported in $[0,1]$, $\rho > 0$ in $(0,1)$ and normalized as to $\int_{\mathbb{R}} \rho = 1$. We define

$$\beta_\epsilon(s) := \frac{1}{2}\int_0^{s/\epsilon} \rho(\tau)d\tau - \frac{1}{2}\int_0^{-s/\epsilon} \rho(\tau)d\tau + \frac{1}{2} + \epsilon,$$

which satisfy the assumptions of Corollary 2.3.

Equations similar to (5) and (6) have been treated in [4, 6]. The solutions of the equations may exhibit a free boundary, whose regularity can be studied with techniques from [1].

Another problem that could be treated with our techniques is

$$\begin{cases} F(D^2u) &= \dfrac{1}{|u|^q} \quad \text{in} \quad \Omega \\ u &= f \quad \text{on} \quad \partial\Omega \end{cases} \tag{7}$$

where $0 < q < 1$ and F is concave. There is a viscosity solution in the pointwise topological sense. Moreover, $u \in C^{1,\frac{1-q}{1+q}}$, the regularity of the first problem (1).

In the proof we use a version of Theorem 2.1 and ideas from the proof of existence of solution to problem (5). Here β_ϵ is the same used to solve (5) and is not monotone.

3. Examples and comparison to our results

In problem (5) we have shown existence of solution to

$$\begin{cases} F(D^2u) &= \dfrac{1}{|u|^q}|\nabla u|^2 \quad \text{in} \quad \Omega \\ u &= f \quad \text{on} \quad \partial\Omega \end{cases}$$

if $q \geq 1$. There is a result in [2] saying that

$$\begin{cases} \Delta u &= \dfrac{1}{|u|^q}|\nabla u|^2 - h \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

has a $\geq 0, \not\equiv 0$ solution if and only if $q \leq 2$, provided $q > 0$ and h is smooth and positive at every compact subset of Ω.

By problem (7) we know that

$$\begin{cases} F(D^2u) &= \dfrac{1}{|u|^q} \quad \text{in} \quad \Omega \\ u &= f \quad \text{on} \quad \partial\Omega \end{cases}$$

has a solution if $0 < q < 1$, remember $f \geq 0$. The first problem (1)

$$\begin{cases} \Delta u &= \dfrac{1}{|u|^q} \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

has no positive solution if $0 < q < 1$. Notice that

$$\begin{cases} -\Delta u = \chi_{\{u>0\}}\big(-u^{-q} + \lambda\big) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has no solution if $q \geq 1$.

But by problem (6)

$$\begin{cases} F(D^2u) & = & \chi_{\{u>0\}} & \text{in} & \Omega \\ u & = & f & \text{on} & \partial\Omega \end{cases}$$

has a solution.

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