São Paulo Journal of Mathematical Sciences **6**, 2 (2012), 203–225

Reduction of order of cluster-type recurrence relations

Inês Cruz

Centro de Matemática da Universidade do Porto, Departamento de Matemática
Faculdade de Ciências da Universidade do Porto, 4169-007 Porto, Portugal

M. Esmeralda Sousa-Dias

Center for Mathematical Analysis, Geometry, and Dynamical Systems
Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

Abstract. Certain nonlinear recurrence relations (of the real line) can be studied within the framework of cluster algebra theory. For this type of relations we develop the tools of Poisson and pre-symplectic structures compatible with a cluster algebra, in order to understand how these structures enable to reduce the recurrence relation to one of lower order. Several examples are worked in detail.

1. Introduction

Cluster algebras have been introduced by Fomin and Zelevinsky in [FoZe02] in order to create an algebraic and combinatorial framework to study total positivity in matrix groups. Since then the theory of cluster algebras has been applied to a wide range of subjects such as: representation theory of quivers; commutative and non-commutative algebraic theory; discrete dynamical systems, etc. In the recent monograph [GeShVa10] the authors survey the advances made in the last decade on the interplay between some cluster algebra structures and Poisson/pre-symplectic structures.

In this work we use the notions of Poisson and pre-symplectic structures compatible with a cluster algebra to reduce the order of (nonlinear) recurrence relations which fit in the framework of cluster algebras. Notably, we address the question of reducing the order of recurrence relations (of order N) of the form

$$x_{n+N}x_n = A^+ + A^-, \quad n = 1, 2, \dots \tag{1}$$

where A^+ and A^- are monomials in the variables x_i (with $i \in I$) and x_j (with $j \in J$) respectively, where $I \cup J = \{n+1, \ldots, n+N-1\}$ and $I \cap J = \emptyset$. Moreover, the (nonnegative) exponents of the monomials A^+ and A^- form an $(N-1)$-tuple which is required to be palindromic. In the theory of cluster algebras the initial vector $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ is called an initial cluster and the x_i's are the cluster variables.

Examples of recurrence relations which can be treated in the context of cluster algebra theory are the following famous Somos-4 and Somos-5 sequences

- Somos-4: $x_{n+4}x_n = x_{n+1}x_{n+3} + x_{n+2}^2$;
- Somos-5: $x_{n+5}x_n = x_{n+1}x_{n+4} + x_{n+2}x_{n+3}$.

These sequences are obtained by iterating a map, which in the case of the Somos-5 sequence is

$$\varphi : (x_1, x_2, x_3, x_4, x_5) \mapsto \left(x_2, x_3, x_4, x_5, \frac{x_2x_5 + x_3x_4}{x_1} \right).$$

There are advantages to study the Somos-4 and 5 sequences using cluster algebra tools. A well known property of these sequences is that they are sequences of integers whenever the N initial points are all equal to 1. Although this property was noticed without using the cluster algebra machinery, it follows as a simple corollary of the famous Laurent phenomenon for cluster algebras (any cluster variable is expressed as a Laurent polynomial of the initial cluster variables with integer coefficients, see [FoZe02a]). Also, using the procedure of reduction of order which will be developed here, it is easy to see that the Somos-5 iteration map can be reduced to a 2-dimensional iteration map belonging to the QRT family of integrable maps (see [QRT] and [Duist10]).

In this work we study recurrence relations of the type (1) using Poisson and pre-symplectic structures compatible with the respective cluster algebra. In particular, we are interested in the question of reducing the order of a given recurrence using either the Poisson approach or the pre-symplectic approach. As we will see, besides the existence of compatible Poisson/pre-symplectic structures with the cluster algebra associated to the sequence, one needs also to take into account the iteration map. The comparison between the Poisson and the pre-symplectic approaches (when both are available) is addressed as well.

The plan of the paper is as follows. In Section 2 we review the essential notions of cluster algebras needed in the following sections. The next two sections are devoted to the study of Poisson and pre-symplectic structures compatible with a given cluster algebra $\mathcal{A}(B)$. We suggest [LibMa] as a general reference for Poisson and pre-symplectic structures. In Section 3 we characterize, in Proposition 1, Poisson structures compatible with a cluster algebra $\mathcal{A}(B)$ defined by a skew-symmetric matrix B. This proposition extends Theorem 4.5 of [GeShVa10] (see also [GeShVa03]) to the case where the matrix B is not of full rank. Propositions 2 and 3 establish the main results needed to reduce the order of a recurrence relation via Poisson structures compatible with the cluster algebra associated to the recurrence. We then apply these results to reduce the order of several recurrence relations of order 6 (Example 4). Pre-symplectic forms compatible with a cluster algebra are characterized in Proposition 4 of the following section, where we obtain a similar result to that in [GeShVa05] (see also Theorem 6.2 of [GeShVa10]). We then use the constructive proof of a theorem of Élie Cartan to explicitly reduce the order of recurrence relations.

2. Preliminaries

A cluster algebra (of geometric type) is described by a pair $(B, \mathbf{x})$, called a *seed*, where:

(1) $B = [b_{ij}]$ is an $N \times n$ ($n \geq N$) matrix whose left $N \times N$ block B_N is a left-skew-symmetrizable matrix (i.e. there exists an $N \times N$ diagonal matrix D such that DB_N is skew-symmetric). The matrix B_N is called the *exchange matrix* and B the *extended exchange* matrix;
(2) The vector $\mathbf{x} = (x_1, \ldots, x_N)$ is called a (initial) cluster and $\tilde{\mathbf{x}} = (x_1, \ldots, x_n)$ an extended cluster;
(3) Both B and $\mathbf{x}$ are subject to *cluster transformations* μ_k (or *mutations in direction* k). These transformations are defined, for each $k = 1, \ldots, N$, by
$$\mu_k(x_i) = x_i' = \begin{cases} x_i & i \neq k \\ \dfrac{\prod_{j:b_{kj}>0} x_j^{b_{kj}} + \prod_{j:b_{kj}<0} x_j^{-b_{kj}}}{x_k}, & i = k. \end{cases} \tag{2}$$
and the transformed matrix $B' = \mu_k(B) = [b'_{ij}]$ is given by
$$b'_{ij} = \begin{cases} -b_{ij}, & (k-i)(l-k) = 0 \\ b_{ij} + \frac{1}{2}\left(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|\right), & \text{otherwise.} \end{cases} \tag{3}$$
Also, if one of the products in (2) is taken over an empty set, then its value is assumed to be 1. We note that μ_k is an involution (i.e. $\mu_k(\mu_k(B)) = B$ and $\mu_k(\mu_k(\mathbf{x})) = \mathbf{x}$);
(4) The variables $x_{N+1}, \ldots, x_n$ are not transformed by μ_k, and so are called *frozen*, or *stable*, variables.

Given an initial seed $(B, \mathbf{x})$ we can apply a transformation μ_k to produce another seed, and then apply another transformation to this seed to produce another, etc. A *cluster algebra* (of geometric type) $\mathcal{A}(B)$ is a subalgebra of the field of rational functions in the cluster variables $x_1, \ldots, x_n$, generated by the union of all clusters.

In this work, the matrix B is assumed to be skew-symmetric (so $n = N$). In particular we are not considering the existence of frozen variables, that is we are only considering what is called a *coefficient free cluster algebra*.

An $N \times N$ skew-symmetric matrix B may be seen as representing a *quiver* (oriented graph) Q with N nodes without loops and 2-cycles, being its entries b_{ij} equal to the number of arrows from i to j. For instance, the following matrix

$$B = \begin{bmatrix} 0 & -r & s & s & -r \\ r & 0 & -r(1+s) & -s(r-1) & s \\ -s & r(1+s) & 0 & -r(1+s) & s \\ -s & s(r-1) & r(1+s) & 0 & -r \\ r & -s & -s & r & 0 \end{bmatrix} \tag{4}$$

represents the quiver in Figure 1.

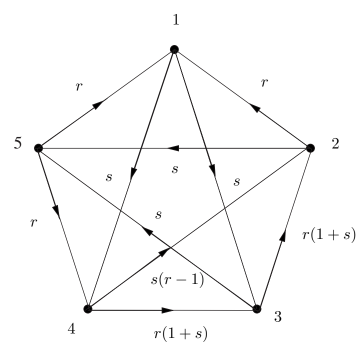

FIGURE 1. Quiver corresponding to the matrix (4) and to the recurrence relation (10).

In terms of the diagram representing the quiver Q, a mutation at the node k corresponds to reversing all the arrows that either originate or terminate at the node k, and all the other arrows are transformed as follows: suppose that in Q there are r arrows from a node i to node j, p arrows from node i to node k and q arrows from node k to node j, then in $Q' = \mu_k(Q)$ we add pq arrows going from node i to node j to the r arrows already there, removing at the same time any two-cycles which might have been created.

Here we consider a particular type of quivers representing recurrence relations of the real line. These quivers were studied in [FoMa] and are designated by *quivers of period 1* (or *1-periodic*). A quiver Q is said to be *of period* m if it satisfies the following chain of mutations

$$Q(1) \stackrel{\mu_1}{\to} Q(2) \stackrel{\mu_2}{\to} Q(3) \cdots \stackrel{\mu_{m-1}}{\to} Q(m) \stackrel{\mu_m}{\to} Q(m+1) = \rho^m Q(1),$$

where $\rho : (1, 2, \dots, N) \mapsto (N, 1, 2, \dots, N-1)$. We remark that if a quiver with N nodes is a period 1 quiver, then it is preserved by the composition $\mu_N \circ \cdots \circ \mu_1$, i.e. $\mu_N \circ \cdots \circ \mu_1(Q) = Q$.

The matrices B representing quivers of period 1 were classified in [FoMa], where it is shown that B is determined by its first row (or column, since B is skew-symmetric) according to the following:

1. The first row $(0, b_{12}, b_{13}, \dots, b_{1N})$ of B must be palindromic, that is
$$b_{1,p+1} = b_{1,N-p+1} \text{ for } p = 1, \dots N-1; \tag{5}$$
2. The other entries of B are given recursively from the entries of the first row by
$$b_{ij} = b_{i-1,j-1} + \epsilon_{i-1,j-1}, \text{ for all } i < j \tag{6}$$
$$\epsilon_{i,j} = \frac{1}{2}\left(b_{1,i+1}\left|b_{1,j+1}\right| - \left|b_{1,i+1}\right| b_{1,j+1}\right); \tag{7}$$
3. The matrix B is symmetric with respect to its anti-diagonal.

The matrix B representing a period 1 quiver with N nodes, also represents a recurrence relation of order N. More precisely, B represents the sequence

$$x_{n+N}x_n = A^+ + A^-, \quad \text{with} \quad A^+ = \prod_{m:b_{1m}>0} x_{n+m}^{b_{1m}}, \quad A^- = \prod_{m:b_{1m}<0} x_{n+m}^{-b_{1m}}, \tag{8}$$

whose iteration map is

$$\varphi : (x_1, x_2, \dots, x_N) \mapsto \left(x_2, x_3, \dots, x_N, \frac{A^+(\hat{x}) + A^-(\hat{x})}{x_1}\right), \tag{9}$$
$$\text{with} \quad \hat{x} = (x_2, \dots, x_N).$$

For instance, the matrix B in (4) satisfies all conditions (5)-(7) and so the quiver in Figure 1 is of period 1. Therefore, this quiver (and so the respective matrix B) represents the recurrence relation

$$x_{n+5}x_n = x_{n+4}^r x_{n+1}^r + x_{n+2}^s x_{n+3}^s. \tag{10}$$

This sequence is precisely the Somos-5 sequence when $r = s = 1$.

Example 1. Examples of recurrence relations of sixth order corresponding to 1-periodic quivers are:

a) $x_{n+6}x_n = x_{n+5}^r x_{n+3}^p x_{n+1}^r + x_{n+2}^s x_{n+4}^s$ with $r, s, p \in \mathbb{N}$.
b) $x_{n+6}x_n = (x_{n+5}x_{n+1})^r + x_{n+2}^s x_{n+3}^p x_{n+4}^s$ with $r, s, p \in \mathbb{N}$.

Using (5)-(7), the matrix B associated to these sequences is

a)

$$B = \begin{bmatrix} 0 & -r & s & -p & s & -r \\ r & 0 & -r(1+s) & s & -p-sr & s \\ -s & r(1+s) & 0 & -r+s(p-r) & s & -p \\ p & -s & r-s(p-r) & 0 & -r(1+s) & s \\ -s & p+sr & -s & r(1+s) & 0 & -r \\ r & -s & p & -s & r & 0 \end{bmatrix}. \qquad (11)$$

b)

$$B = \begin{bmatrix} 0 & -r & s & p & s & -r \\ r & 0 & -r(1+s) & s-pr & p-sr & s \\ -s & r(1+s) & 0 & -r(1+s) & s-pr & p \\ -p & -s+pr & r(1+s) & 0 & -r(1+s) & s \\ -s & -p+sr & -s+pr & r(1+s) & 0 & -r \\ r & -s & -p & -s & r & 0 \end{bmatrix}. \qquad (12)$$

3. Compatible Poisson Structures

Poisson structures compatible with a cluster algebra were introduced in [GeShVa03] (see also [GeShVa10]) and have been applied, for instance, to Grassmannians in [GeShStVa], to directed networks (in [GeShVa09] and [GeShVa12]) and even to the theory of integrable systems (Toda flows in GL_n) in [GeShVa11].

A Poisson structure is compatible with a cluster algebra if in any set of cluster variables, the Poisson bracket is given by the simplest possible kind of homogeneous quadratic bracket.

Let $(B, \mathbf{x})$ be an initial seed, where B is an $N \times N$ integer skew-symmetric matrix and $\mathbf{x} = (x_1, \ldots, x_N)$. A (nontrivial) Poisson bracket $\{,\}$ is *compatible with* $\mathcal{A}(B)$ if:

- it is of the form
$$\{x_i, x_j\} = c_{ij}x_ix_j \qquad (13)$$
with $C = [c_{ij}]$ a (nontrivial) integer skew-symmetric matrix; and for $\mathbf{x}' = (x'_1, \ldots, x'_n)$ given by (2), one has again
$$\{x'_i, x'_j\} = c'_{ij}x'_ix'_j$$

with $C' = [c'_{ij}]$ an integer skew-symmetric matrix.

Note that any skew-symmetric matrix C defines a Poisson bracket through (13). The matrix C is called the *coefficient matrix* of the bracket with respect to the variables $(x_1, \dots, x_N)$.

In the cluster algebra literature, a Poisson bracket of the type (13) is known as a *log-canonical Poisson bracket* (or the cluster variables are referred to as a *log-canonical system of coordinates*), since in coordinates $z_i = \log x_i$ the Poisson bracket has the *canonical* form

$$\{z_i, z_j\} = c_{ij}.$$

Poisson brackets compatible with a cluster algebra $\mathcal{A}(B)$ were characterized in Theorem 4.5 of [GeShVa10] (see also [GeShVa03]) for the case of full rank matrices B. The next proposition extends this theorem to any skew-symmetric matrix B. We remark that singular matrices B are particularly relevant in recurrence relations whose order can be lowered.

Proposition 1. *Let $(B, \mathbf{x})$ be an initial seed, with B an $N \times N$ integer skew-symmetric matrix and $\mathbf{x} = (x_1, \dots, x_N)$. Then a Poisson structure (given by a matrix C) is compatible with the cluster algebra $\mathcal{A}(B)$ if and only if BC is a diagonal matrix D.*

In particular, if B is invertible then C can be chosen to be invertible.

Proof. Consider a Poisson structure given by (13) and a mutation in any direction k. In the mutated variables $(x'_1, \dots, x'_N)$ the Poisson structure must satisfy

$$\{x'_i, x'_j\} = c'_{ij} x'_i x'_j$$

which is clearly true (with $c'_{ij} = c_{ij}$) if i, j are both different from k. If $i = k$ (and $j \neq k$), then $x'_j = x_j$ and $x'_i = x'_k = \frac{A^+(\hat{x}) + A^-(\hat{x})}{x_k}$, with

$$A^+(\hat{x}) = \prod_{m: b_{km} > 0} x_m^{b_{km}}, \quad A^-(\hat{x}) = \prod_{m: b_{km} < 0} x_m^{-b_{km}},$$

and $\hat{x} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N)$. Then

$$\begin{aligned}
\{x'_k, x'_j\} &= \left\{ \frac{A^+(\hat{x}) + A^-(\hat{x})}{x_k}, x_j \right\} \\
&= -\frac{1}{x_k^2} \left(A^+(\hat{x}) + A^-(\hat{x})\right) \{x_k, x_j\} \\
&\quad + \frac{1}{x_k} \left[\sum_{l:b_{kl}>0} \frac{b_{kl}}{x_l} A^+(\hat{x}) \{x_l, x_j\} - \sum_{l:b_{kl}<0} \frac{b_{kl}}{x_l} A^-(\hat{x}) \{x_l, x_j\} \right] \\
&= -c_{kj} x'_k x'_j + \frac{x'_j x'_k}{A^+(\hat{x}') + A^-(\hat{x}')} \left[A^+(\hat{x}') \sum_{l:b_{kl}>0} b_{kl} c_{lj} - A^-(\hat{x}') \sum_{l:b_{kl}<0} b_{kl} c_{lj} \right].
\end{aligned}$$

Therefore, $\{x'_k, x'_j\} = c'_{kj} x'_k x'_j$ if and only if

$$\sum_{l:b_{kl}>0} b_{kl} c_{lj} = -\sum_{l:b_{kl}<0} b_{kl} c_{lj} \Longleftrightarrow (BC)_{kj} - b_{kk} c_{kj} = 0.$$

As B is skew-symmetric, the above condition is equivalent to $(BC)_{kj} = 0$, for $j \neq k$. Since k is arbitrary, this amounts to $BC = D$ with D a diagonal matrix.

If B is invertible then the condition $BC = D$ is equivalent to $C = B^{-1}D$, with D diagonal. In particular we can choose D to be invertible and so will be C. $\square$

Remark 1. If the matrix B is singular, a compatible Poisson structure might not exist. For instance, it is easy to see that the cluster algebra defined by the matrix

$$B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \tag{14}$$

(corresponding to a cyclic quiver of three nodes), does not admit a nontrivial compatible Poisson structure.

Example 2. A cluster algebra $\mathcal{A}(B)$ can admit compatible Poisson structures of different ranks. For instance, for the following matrix B we get a two parameter family of Poisson brackets with coefficient matrices $C_{a,b}$.

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad C_{a,b} = \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ -a & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \end{bmatrix}. \tag{15}$$

We also remark that the above matrix B corresponds to a quiver of period 1, more precisely to what is called a *primitive quiver* (such quivers were used in [FoMa] as the building blocks of 1-periodic quivers). This matrix B represents the sequence

$$x_{n+4}x_n = x_{n+2} + 1. \tag{16}$$

3.1. **Reduction of Order by Compatible Poisson Structures.** For the purpose of *reduction of* (8) to a lower order recurrence relation, it is important to consider a special kind of compatible Poisson structures.

Let φ be the iteration map (9) corresponding to the recurrence relation (8). To reduce (8) to a recurrence relation of order $k < N$ it is important to find a set of k functions, $y_1, \ldots, y_k$, which is *invariant under* φ, meaning that

$$y_i \circ \varphi = F_i(y_1, \ldots, y_k), \quad i = 1, \ldots, k.$$

This can be achieved if:

- φ is a *Poisson map*, i.e, for all functions f and g
$$\{f \circ \varphi, g \circ \varphi\} = \{f, g\} \circ \varphi;$$
- each y_i is a *Casimir*, that is its bracket with any other function vanishes:
$$\{y_i, f\} = 0, \quad i = 1, \ldots, k.$$

In such case the set of Casimirs will be invariant under φ.

The next proposition characterizes compatible Poisson structures with the cluster algebra $\mathcal{A}(B)$ for which the iteration map φ is a Poisson map.

Proposition 2. *Consider the recurrence relation* (8) *and the matrix* B *representing it. Let*

$$\{x_i, x_j\} = c_{ij}x_ix_j$$

be a Poisson bracket which is compatible with the cluster algebra $\mathcal{A}(B)$ *(i.e.,* BC *is diagonal). Then the iteration map* φ *in* (9) *is a Poisson map if and only if:*

- $C = [c_{ij}]$ *is a band matrix (in the sense* $c_{ij} = c_{i+1,j+1}$*)*
- C *satisfies*

$$c_{iN} = c_{1,i+1} - \sum_{l:b_{1l}>0} b_{1l}c_{l,i+1}, \quad i \in \{1, \ldots, N-1\}. \tag{17}$$

Proof. φ is a Poisson map if and only if

$$\{x_i, x_j\} \circ \varphi = \{x_i \circ \varphi, x_j \circ \varphi\}, \quad \forall i, j \in \{1, \ldots, N\}.$$

If $i \neq N$ and $j \neq N$ then

$$\{x_i, x_j\} \circ \varphi = (c_{ij}x_ix_j) \circ \varphi = c_{ij}x_{i+1}x_{j+1}$$
$$\{x_i \circ \varphi, x_j \circ \varphi\} = \{x_{i+1}, x_{j+1}\} = c_{i+1,j+1}x_{i+1}x_{j+1},$$

so necessarily $c_{ij} = c_{i+1,j+1}$ for all $i, j \in \{1, \dots, N-1\}$. This means that C is a band matrix.

If $j = N$, then we must impose

$$\{x_i, x_N\} \circ \varphi = \{x_i \circ \varphi, x_N \circ \varphi\}, \quad i \in \{1, \dots, N-1\}. \tag{18}$$

In this case

$$\{x_i, x_N\} \circ \varphi = c_{iN} x_{i+1} \frac{A^+(\hat{x}) + A^-(\hat{x})}{x_1},$$

whereas

$$\begin{aligned}
\{x_i \circ \varphi, x_N \circ \varphi\} &= \left\{ x_{i+1}, \frac{A^+(\hat{x}) + A^-(\hat{x})}{x_1} \right\} \\
&= -\frac{A^+(\hat{x}) + A^-(\hat{x})}{x_1^2} \{x_{i+1}, x_1\} \\
&\quad + \frac{1}{x_1} \left[\sum_{k:b_{1k}>0} b_{1k} \frac{A^+(\hat{x})}{x_k} \{x_{i+1}, x_k\} - \sum_{k:b_{1k}<0} b_{1k} \frac{A^-(\hat{x})}{x_k} \{x_{i+1}, x_k\} \right] \\
&= -c_{i+1,1} x_{i+1} \frac{A^+(\hat{x}) + A^-(\hat{x})}{x_1} \\
&\quad + \frac{x_{i+1}}{x_1} \left[A^+(\hat{x}) \sum_{k:b_{1k}>0} b_{1k} c_{i+1,k} - A^-(\hat{x}) \sum_{k:b_{1k}<0} b_{1k} c_{i+1,k} \right] \\
&= -c_{i+1,1} x_{i+1} \frac{A^+(\hat{x}) + A^-(\hat{x})}{x_1} \\
&\quad + \frac{x_{i+1}}{x_1} \left[-A^+(\hat{x}) \sum_{k:b_{1k}>0} b_{1k} c_{k,i+1} - A^-(\hat{x}) \sum_{k:b_{1k}>0} b_{1k} c_{k,i+1} \right].
\end{aligned}$$

In the last identity we used skew-symmetry of C and the fact that BC is diagonal, which translates into

$$\sum_{k:b_{1k}<0} b_{1k} c_{k,i+1} = -\sum_{k:b_{1k}>0} b_{1k} c_{k,i+1}.$$

Finally (18) holds if and only if

$$c_{iN} = c_{1,i+1} - \sum_{k:b_{1k}>0} b_{1k} c_{k,i+1}, \quad i \in \{1, \dots, N-1\}. \tag{19}$$

$\square$

We remark that the condition of φ being a Poisson map imposes strong conditions on the number of Poisson structures compatible with $\mathcal{A}(B)$. For example, for the recurrence relation (16), φ is a Poisson map for $C_{a,b}$ in (15) if and only if $a = b$.

Example 3. Consider the recurrence relation (10) with $r = s = 1$, corresponding to the Somos-5 sequence. The corresponding matrix B has rank 2 and there is just one (up to a constant) compatible Poisson bracket for which the iteration map is Poisson. The respective coefficient matrix of this bracket is

$$C = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 & 3 \\ -2 & -1 & 0 & 1 & 2 \\ -3 & -2 & -1 & 0 & 1 \\ -4 & -3 & -2 & -1 & 0 \end{bmatrix}. \tag{20}$$

Example 4. Consider the recurrence relations given in Example 1.

a1) The matrix B in Example 1-a) with $p = 4, r = 2$ and $s = 6$ has rank two, and there are two compatible Poisson brackets for which the map φ is a Poisson map. These brackets are given by the matrices

$$C_1 = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & -1 & -3 & -7 & -18 \\ 0 & 0 & 0 & -1 & -3 & -7 \\ 1 & 0 & 0 & 0 & -1 & -3 \\ 3 & 1 & 0 & 0 & 0 & -1 \\ 7 & 3 & 1 & 0 & 0 & 0 \\ 18 & 7 & 3 & 1 & 0 & 0 \end{bmatrix}. \tag{21}$$

a2) The matrix B in Example 1-a) with $p = 4, r = 2$ and $s = 3$ has rank four, and there is only one compatible Poisson bracket for which the map φ is a Poisson map. The coefficient matrix for this bracket is C_1 defined in (21).

b1) In Example 1-b) with $p = 4, r = 2$ and $s = 2$, the matrix B has rank two, and there are two compatible Poisson brackets for which the map φ is a Poisson map. The respective coefficient matrices are:

$$C_1' = \begin{bmatrix} 0 & 1 & 0 & 3 & 4 & 9 \\ -1 & 0 & 1 & 0 & 3 & 4 \\ 0 & -1 & 0 & 1 & 0 & 3 \\ -3 & 0 & -1 & 0 & 1 & 0 \\ -4 & -3 & 0 & -1 & 0 & 1 \\ -9 & -4 & -3 & 0 & -1 & 0 \end{bmatrix}, \quad C_2' = \begin{bmatrix} 0 & 0 & 1 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ -1 & 0 & 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ -3 & -1 & -1 & 0 & 0 & 0 \\ -6 & -3 & -1 & -1 & 0 & 0 \end{bmatrix}. \tag{22}$$

Summarizing, for the recurrence relations

a1) $x_{n+6}x_n = (x_{n+5}x_{n+1})^2 x_{n+3}^4 + (x_{n+2}x_{n+4})^6$
a2) $x_{n+6}x_n = (x_{n+5}x_{n+1})^2 x_{n+3}^4 + (x_{n+2}x_{n+4})^3$
b1) $x_{n+6}x_n = (x_{n+5}x_{n+1})^2 + (x_{n+2}x_{n+4})^2 x_{n+3}^4$

we have

	rank B	rank C	
a1)	rank $B = 2$	rank $C_1 = 2$	rank $C_2 = 4$
a2)	rank $B = 4$	rank $C_1 = 2$	
b1)	rank $B = 2$	rank $C_1' = 4$	rank $C_2' = 4$

To lower the order of a recurrence relation one should now find Casimirs for the compatible Poisson brackets. *Laurent monomials* which are Casimirs can be computed directly from the matrix C as stated in the following proposition.

Proposition 3. *Let* $\mathbf{v} = (v_1, \ldots, v_N) \in \mathbb{Z}^N$ *and* $\mathbf{x^v} = x_1^{v_1} x_2^{v_2} \cdots x_N^{v_N}$. *Then* $\mathbf{x^v}$ *is a Casimir of the Poisson bracket*

$$\{x_i, x_j\} = c_{ij} x_i x_j$$

if and only if $\mathbf{v} \in \ker C$.

Proof. For any $i \in \{1, \ldots, N\}$ we have

$$\{x_i, \mathbf{x^v}\} = \sum_{k=1}^{N} v_k c_{ik} \mathbf{x^v} x_i = \mathbf{x^v} x_i \sum_{k=1}^{N} c_{ik} v_k = \mathbf{x^v} x_i \left(C\mathbf{v}\right)_i .$$

Thus $\mathbf{x^v}$ is a Casimir if and only if $C\mathbf{v} = \mathbf{0}$. □

Example 5. For the Somos-5 recurrence relation, the kernel of the matrix C in (20) is

$$\ker C = \langle (1, -2, 1, 0, 0), (0, 1, -2, 1, 0), (0, 0, 1, -2, 1) \rangle,$$

giving the following Casimirs

$$y_1 = \frac{x_1 x_3}{x_2^2}, \quad y_2 = \frac{x_2 x_4}{x_3^2}, \quad y_3 = \frac{x_3 x_5}{x_4^2}.$$

These Casimirs satisfy the relations $y_i \circ \varphi = y_{i+1}$ for $i = 1, 2$ and

$$y_3 \circ \varphi = \frac{1 + y_2 y_3}{y_1 y_2^2 y_3^2},$$

showing that the fifth order recurrence relation can be reduced to a third order relation whose iteration map is

$$\hat{\varphi}(y_1, y_2, y_3) = \left(y_2, y_3, \frac{1 + y_2 y_3}{y_1 y_2^2 y_3^2} \right). \tag{23}$$

Example 6. For the coefficient matrices in Example 4 we have

$$\ker C_1 = \langle (1,0,1,0,0,0),(0,1,0,1,0,0),(0,0,1,0,1,0),(0,0,0,1,0,1)\rangle \tag{24}$$

$$\ker C_2 = \langle (1,-3,2,-3,1,0),(0,1,-3,2,-3,1)\rangle \tag{25}$$

$$\ker C_1' = \langle (1,-1,-2,-1,1,0),(0,1,-1,-2,-1,1)\rangle \tag{26}$$

$$\ker C_2' = \ker C_1'. \tag{27}$$

a1) Casimirs for the Poisson structure defined by C_1 are

$$y_1 = x_1x_3, \quad y_2 = x_2x_4, \quad y_3 = x_3x_5, \quad y_4 = x_4x_6. \tag{28}$$

These Casimirs are such that $y_i \circ \varphi = y_{i+1}$ for $i = 1,2,3$ and

$$y_4 \circ \varphi = \frac{y_3}{y_1}\left(y_2^2y_4^2 + y_3^6\right).$$

This shows that the sixth order recurrence relation can be reduced to a fourth order relation whose iteration map is

$$\hat{\varphi}(y_1,y_2,y_3,y_4) = \left(y_2,y_3,y_4,\frac{y_2^2y_3y_4^2 + y_3^7}{y_1}\right). \tag{29}$$

Using the Poisson structure defined by C_2 instead of C_1 we have the Casimirs:

$$z_1 = \frac{x_1x_3^2x_5}{x_2^3x_4^3}, \quad z_2 = \frac{x_2x_4^2x_6}{x_3^3x_5^3}.$$

As $z_1 \circ \varphi = z_2$ and

$$z_2 \circ \varphi = \frac{1+z_2^2}{z_1z_2^3},$$

the reduced relation is now a second order relation whose iteration map is given by

$$\tilde{\varphi}(z_1,z_2) = \left(z_2,\frac{1+z_2^2}{z_1z_2^3}\right). \tag{30}$$

a2) Casimirs for C_1 are given in (28), but now the reduced iteration map is

$$\hat{\varphi}(y_1,y_2,y_3,y_4) = \left(y_2,y_3,y_4,\frac{y_2^2y_3y_4^2 + y_3^4}{y_1}\right). \tag{31}$$

b1) Finally, the Casimirs for both Poisson structures defined by C_1' and C_2' are

$$y_1 = \frac{x_1x_5}{x_2x_3^2x_4}, \quad y_2 = \frac{x_2x_6}{x_3x_4^2x_5}.$$

The reduced relation has iteration map given by

$$\hat{\varphi}(y_1, y_2) = \left(y_2, \frac{1 + y_2^2}{y_1 y_2}\right). \tag{32}$$

Remark 2. It is worth noting that the fourth order relation (29) could be further reduced to the second order relation (30). In fact, considering the projection

$$\pi(y_1, y_2, y_3, y_4) = \left(\frac{y_1 y_3}{y_2^3}, \frac{y_2 y_4}{y_3^3}\right)$$

we have

$$\pi \circ \hat{\varphi} = \tilde{\varphi} \circ \pi.$$

4. Compatible Pre-symplectic Structures

Compatible pre-symplectic structures with a given cluster algebra can also be used to reduce the order of recurrence relations of the form (8). They were introduced in [GeShVa05] (see also [GeShVa10]) as an alternative tool when there are no (nontrivial) Poisson structures compatible with $\mathcal{A}(B)$, such as the case in Remark 1.

Let $(B, \mathbf{x})$ be an initial seed, with B an $N \times N$ integer skew-symmetric matrix and $\mathbf{x} = (x_1, \ldots, x_N)$. A closed 2-form ω (i.e., a pre-symplectic structure) is *compatible with* $\mathcal{A}(B)$ if:

- it is of the form

$$\omega = \sum_{i<j} \omega_{ij} \frac{dx_i}{x_i} \wedge \frac{dx_j}{x_j}, \tag{33}$$

  for some integers ω_{ij};
- for $\mathbf{x}' = (x'_1, \ldots, x'_N)$ given by (2), one has again

$$\omega = \sum_{i<j} \omega'_{ij} \frac{dx'_i}{x'_i} \wedge \frac{dx'_j}{x'_j},$$

  for some integers ω'_{ij}.

Such a pre-symplectic form is usually given by the integer skew-symmetric matrix $\Omega = [\omega_{ij}]$, which is called the *coefficient matrix* of the form with respect to the variables $(x_1, \ldots, x_N)$.

Just as in the Poisson situation, a pre-symplectic form of the type (33) is known as a *log-canonical pre-symplectic structure* since in coordinates

$z_i = \log x_i$ it has the *canonical* form

$$\omega = \sum_{i<j} \omega_{ij} dz_i \wedge dz_j.$$

The next proposition characterizes pre-symplectic forms compatible with a cluster algebra. The contents of this proposition coincide with that of Theorem 6.2 in [GeShVa10] when the matrix B has no zero rows.

Proposition 4. *Let $(B, \mathbf{x})$ be an initial seed, with B an $N \times N$ integer skew-symmetric matrix and $\mathbf{x} = (x_1, \dots, x_N)$.*

Then a pre-symplectic structure (given by a skew-symmetric integer matrix Ω) is compatible with the cluster algebra $\mathcal{A}(B)$ if and only if the k^{th} row of Ω is a multiple of the k^{th} row of B, whenever the latter is nonzero.

In particular, if there is one row of B with $N-1$ nonzero entries, then Ω is a multiple of B.

Proof. Let ω be a pre-symplectic form given by (33) and consider a mutation in any direction k. Then, with $\hat{x} = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_N)$ and

$$A^+(\hat{x}) = \prod_{m:b_{km}>0} x_m^{b_{km}}, \quad A^-(\hat{x}) = \prod_{m:b_{km}<0} x_m^{-b_{km}},$$

we have $\dfrac{dx_i}{x_i} = \dfrac{dx'_i}{x'_i}$ for $i \neq k$, and

$$\frac{dx_k}{x_k} = -\frac{dx'_k}{x'_k}$$

$$+ \overbrace{\frac{1}{A^+(\hat{x}') + A^-(\hat{x}')} \left(A^+(\hat{x}') \sum_{m:b_{km}>0} b_{km} \frac{dx'_m}{x'_m} - A^-(\hat{x}') \sum_{m:b_{km}<0} b_{km} \frac{dx'_m}{x'_m} \right)}^{\alpha},$$

so that

$$\begin{aligned} \sum_{i<j} \omega_{ij} \frac{dx_i}{x_i} \wedge \frac{dx_j}{x_j} &= \sum_{k \neq i < j \neq k} \omega_{ij} \frac{dx'_i}{x'_i} \wedge \frac{dx'_j}{x'_j} + \sum_{i<k} \omega_{ik} \frac{dx'_i}{x'_i} \wedge \left(-\frac{dx'_k}{x'_k} + \alpha \right) + \\ &+ \sum_{j>k} \omega_{kj} \left(-\frac{dx'_k}{x'_k} + \alpha \right) \wedge \frac{dx'_j}{x'_j}. \end{aligned}$$

If ω is compatible with $\mathcal{A}(B)$ then:

- for $i < k$: $\omega'_{ik} = -\omega_{ik}$
- for $j > k$: $\omega'_{kj} = -\omega_{kj}$

- for $i < j < k$:

$$\omega'_{ij} = \omega_{ij} + \begin{cases} \frac{A^+\left(\omega_{ik}b_{kj} - \omega_{jk}b_{ki}\right)}{A^+ + A^-}, & \text{if } b_{ki} > 0,\ b_{kj} > 0 \\ -\frac{A^-\left(\omega_{ik}b_{kj} - \omega_{jk}b_{ki}\right)}{A^+ + A^-}, & \text{if } b_{ki} < 0,\ b_{kj} < 0 \\ \frac{A^+\omega_{ik}b_{kj} + A^-\omega_{jk}b_{ki}}{A^+ + A^-}, & \text{if } b_{ki} < 0,\ b_{kj} > 0 \\ -\frac{A^-\omega_{ik}b_{kj} + A^+\omega_{jk}b_{ki}}{A^+ + A^-}, & \text{if } b_{ki} > 0,\ b_{kj} < 0 \end{cases}$$

These conditions define integers ω'_{ij} if and only if

$$\omega_{ik}b_{kj} - \omega_{jk}b_{ki} = 0, \quad i < j < k \tag{34}$$

- for $i < k < j$:

$$\omega'_{ij} = \omega_{ij} + \begin{cases} \frac{A^+\left(\omega_{ik}b_{kj} + \omega_{kj}b_{ki}\right)}{A^+ + A^-}, & \text{if } b_{ki} > 0,\ b_{kj} > 0 \\ -\frac{A^-\left(\omega_{ik}b_{kj} + \omega_{kj}b_{ki}\right)}{A^+ + A^-}, & \text{if } b_{ki} < 0,\ b_{kj} < 0 \\ \frac{A^+\omega_{ik}b_{kj} - A^-\omega_{kj}b_{ki}}{A^+ + A^-}, & \text{if } b_{ki} < 0,\ b_{kj} > 0 \\ -\frac{A^-\omega_{ik}b_{kj} - A^+\omega_{kj}b_{ki}}{A^+ + A^-}, & \text{if } b_{ki} > 0,\ b_{kj} < 0 \end{cases}$$

This amounts to the conditions

$$\omega_{ik}b_{kj} + \omega_{kj}b_{ki} = 0, \quad i < k < j \tag{35}$$

- for $k < i < j$:

$$\omega'_{ij} = \omega_{ij} + \begin{cases} \frac{A^+\left(-\omega_{ki}b_{kj} + \omega_{kj}b_{ki}\right)}{A^+ + A^-}, & \text{if } b_{ki} > 0,\ b_{kj} > 0 \\ \frac{A^-\left(\omega_{ki}b_{kj} - \omega_{kj}b_{ki}\right)}{A^+ + A^-}, & \text{if } b_{ki} < 0,\ b_{kj} < 0 \\ -\frac{A^+\omega_{ki}b_{kj} + A^-\omega_{kj}b_{ki}}{A^+ + A^-}, & \text{if } b_{ki} < 0,\ b_{kj} > 0 \\ \frac{A^-\omega_{ki}b_{kj} + A^+\omega_{kj}b_{ki}}{A^+ + A^-}, & \text{if } b_{ki} > 0,\ b_{kj} < 0 \end{cases}$$

In this last case we have the conditions

$$\omega_{ki}b_{kj} - \omega_{kj}b_{ki} = 0, \quad k < i < j. \tag{36}$$

To conclude the proof we observe that, if the k^{th} row of B is nonzero then there is $m \neq k$ such that $b_{km} \neq 0$. In this case conditions (34), (35) and (36) guarantee that, for all $i \in \{1, \ldots, N\}$

$$\omega_{ki} = \lambda_k b_{ki} \quad \text{with} \quad \lambda_k = \frac{\omega_{km}}{b_{km}}$$

(the cases $i = k$ and $i = m$ are trivially satisfied). This is equivalent to the k^{th} row of Ω being a multiple of the k^{th} row of B.

If the k^{th} row of B is zero then the k^{th} row of Ω is arbitrary because (34), (35) and (36) hold trivially.

Finally observe that, if the k^{th} row of B has only one zero entry (i.e. b_{kk}), then the skew-symmetry of both B and Ω assure that $\lambda_i = \lambda_k$ for all $i \neq k$ and therefore $\Omega = \lambda B$.

□

Remark 3. Contrary to the Poisson situation, a pre-symplectic structure compatible with $\mathcal{A}(B)$ always exists since we can choose $\Omega = B$. This choice for ω will be referred to as *standard pre-symplectic structure*. For example, if B is the matrix (14) in Remark 1 then

$$\omega = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} - \frac{dx_1}{x_1} \wedge \frac{dx_3}{x_3} + \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3}$$

is the standard pre-symplectic structure compatible with $\mathcal{A}(B)$.

Example 7. Consider the matrix

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Then the pre-symplectic form

$$\omega = a\frac{dx_1}{x_1} \wedge \frac{dx_4}{x_4} + b\frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3}$$

is compatible with $\mathcal{A}(B)$, for all $a, b \in \mathbb{Z}$.

4.1. **Reduction of Order by Compatible Pre-symplectic Structures.** Like in the Poisson approach, compatible pre-symplectic structures can be used to lower the order of the recurrence (8). Again, in order to lower the order of the recurrence it is important to find $2k < N$ functions $y_1, \ldots, y_{2k}$ which form an invariant set under φ.

In [FoHo11] and [FoHo12], the standard pre-symplectic structure has been used to lower the order of recurrence relations of the form (8) in the case B has rank two. Here, however, we will follow a different strategy, which applies easily to the case where B has rank $2k > 2$. To be more precise we observe that, if

- φ preserves ω, i.e., $\varphi^*\omega = \omega$;
- $y_1, \dots, y_{2k}$ are such that

$$\omega = dy_1 \wedge dy_2 + \cdots + dy_{2k-1} \wedge dy_{2k}$$

then the set of functions $y_1, \dots, y_{2k}$ is invariant under φ. The first of these conditions is guaranteed to hold in the case ω is the standard pre-symplectic structure, by Lemma 2.3 in [FoHo12]. The existence of $y_1, \dots, y_k$ comes from the following theorem of E. Cartan, whose constructive proof can be found, for instance, in [LibMa].

Theorem 1 (E. Cartan)**.** *Let ω be a 2-form on a linear vector space V such that $rank(\omega) = 2p$. Then, there are linear coordinates $f^1, \dots, f^{2p} \in V^*$ such that*

$$\omega = f^1 \wedge f^2 + \cdots + f^{2p-1} \wedge f^{2p}. \tag{37}$$

Proof. Let $\{e_1, \dots, e_N\}$ be an arbitrary basis for V and denote by $\{e^1, \dots, e^N\}$ the linear coordinates on this basis. Then

$$\omega = \frac{1}{2}\sum_{i,j} \omega_{ij} e^i \wedge e^j, \qquad \text{with} \quad \omega_{ji} = -\omega_{ij}.$$

Reordering if necessary the elements in the original basis we can assume that $\omega_{12} \neq 0$. Define

$$f^1 = \frac{1}{\omega_{12}} \sum_{k=1}^{N} \omega_{1k} e^k, \quad f^2 = \sum_{l=1}^{N} \omega_{2l} e^l, \tag{38}$$

so that

$$f^1 \wedge f^2 = \omega_{12} e^1 \wedge e^2 + \sum_{i=3}^{N} \omega_{1i} e^1 \wedge e^i + \sum_{j=3}^{N} \omega_{2j} e^2 \wedge e^j + \alpha$$

with α depending only on $\{e^3, \dots, e^N\}$.

The 2-form

$$\tilde{\omega} = \omega - f^1 \wedge f^2$$

is a 2-form on the $(N-2)$-dimensional vector space with basis $\{e_3, \dots, e_N\}$. Furthermore $rank(\tilde{\omega}) = 2p - 2$.

If $rank(\omega) = 2$ then $\tilde{\omega} \equiv 0$ and the proof is finished since $\omega = f^1 \wedge f^2$. Otherwise, the previous procedure is repeated now for $\tilde{\omega}$ instead of ω. The decomposition will be obtained in p steps. $\square$

Remark 4. The decomposition (37) is clearly not unique. For example, if $\omega = f^1 \wedge f^2$, then also $\omega = f^1 \wedge \left(f^2 + af^1\right)$, $a \in \mathbb{R}$. This fact will be used in the reduction of the recurrence relations.

Example 8. Let us consider the Somos-5 sequence defined by the matrix B in (4) with $r = s = 1$. That is,

$$B = \begin{bmatrix} 0 & -1 & 1 & 1 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ -1 & 2 & 0 & -2 & 1 \\ -1 & 0 & 2 & 0 & -1 \\ 1 & -1 & -1 & 1 & 0 \end{bmatrix}.$$

Consider the (standard) pre-symplectic form $\Omega = B$. Using (38) we get directly from the 1st and 2nd rows of B:

$$\omega = d\left(\underbrace{z_2 - z_3 - z_4 + z_5}_{f^1}\right) \wedge d\left(\underbrace{z_1 - 2z_3 + z_5}_{f^2}\right).$$

Replacing f^2 by $f^2 - f^1$, we obtain

$$\omega = d\left(z_2 - z_3 - z_4 + z_5\right) \wedge d\left(z_1 - z_2 - z_3 + z_4\right).$$

Written in terms of the x_i-coordinates we have

$$\omega = d\log\left(\frac{x_2x_5}{x_3x_4}\right) \wedge d\log\left(\frac{x_1x_4}{x_2x_3}\right).$$

Taking

$$y_1 = \frac{x_1x_4}{x_2x_3}, \quad y_2 = \frac{x_2x_5}{x_3x_4},$$

the reduced iteration map is

$$\hat{\varphi}(y_1, y_2) = \left(y_2, \frac{1 + y_2}{y_1y_2}\right).$$

We remark that $\hat{\varphi}$ belongs to the QRT family of integrable maps (see [QRT]). This same reduced iteration map has been obtained in [FoHo11] by following a different procedure.

Example 9. Consider again the recurrence relation of Example 4-a1), where

$$B = \begin{bmatrix} 0 & -2 & 6 & -4 & 6 & -2 \\ 2 & 0 & -14 & 6 & -16 & 6 \\ -6 & 14 & 0 & 10 & 6 & -4 \\ 4 & -6 & -10 & 0 & -14 & 6 \\ -6 & 16 & -6 & 14 & 0 & -2 \\ 2 & -6 & 4 & -6 & 2 & 0 \end{bmatrix}.$$

We consider the pre-symplectic form $\Omega = \frac{1}{2}B$ which is compatible with $\mathcal{A}(B)$. This form is given in coordinates $z_i = \log x_i$ by

$$\omega = \sum_{i<j} \frac{b_{ij}}{2} dz_i \wedge dz_j,$$

and has rank 2. Using (38), the 1st and 2nd rows of Ω give again

$$\omega = d\left(\underbrace{z_2 - 3z_3 + 2z_4 - 3z_5 + z_6}_{f^1}\right) \wedge d\left(\underbrace{z_1 - 7z_3 + 3z_4 - 8z_5 + 3z_6}_{f^2}\right),$$

or, replacing f^2 by $f^2 - 3f^1$:

$$\omega = d\left(z_2 - 3z_3 + 2z_4 - 3z_5 + z_6\right) \wedge d\left(z_1 - 3z_2 + 2z_3 - 3z_4 + z_5\right).$$

Written in terms of the x_i-coordinates this amounts to

$$\omega = d\log\left(\frac{x_2x_4^2x_6}{x_3^3x_5^3}\right) \wedge d\log\left(\frac{x_1x_3^2x_5}{x_2^3x_4^3}\right),$$

producing, as in (30), a reduced recurrence relation whose iteration map is the symplectic map

$$\hat{\varphi}(y_1, y_2) = \left(y_2, \frac{1+y_2^2}{y_1y_2^3}\right)$$

where

$$y_1 = \frac{x_1x_3^2x_5}{x_2^3x_4^3}, \quad y_2 = \frac{x_2x_4^2x_6}{x_3^3x_5^3}.$$

This reduced relation has also been obtained in [FoHo11].

The recurrence considered in Example 4-b1) is treated in a completely analogous way, using the pre-symplectic form $\Omega = \frac{1}{2}B$. The respective reduced iteration map is precisely (32).

We proceed with an example where B has rank four, as this situation requires going one step further in Cartan's decomposition.

Example 10. Consider the recurrence relation in Example 4-a2) given by

$$B = \begin{bmatrix} 0 & -2 & 3 & -4 & 3 & -2 \\ 2 & 0 & -8 & 3 & -10 & 3 \\ -3 & 8 & 0 & 4 & 3 & -4 \\ 4 & -3 & -4 & 0 & -8 & 3 \\ -3 & 10 & -3 & 8 & 0 & -2 \\ 2 & -3 & 4 & -3 & 2 & 0 \end{bmatrix}.$$

Consider the pre-symplectic form $\Omega = 2B$ which in coordinates $z_i = \log x_i$ is the form

$$\omega = \sum_{i<j} 2b_{ij} dz_i \wedge dz_j.$$

Using (38) we arrive at

$$df^1 = d\left(z_2 - \frac{3}{2}z_3 + 2z_4 - \frac{3}{2}z_5 + z_6\right),$$
$$df^2 = d\left(4z_1 - 16z_3 + 6z_4 - 20z_5 + 6z_6\right).$$

Because $rank(\omega) = 4$ we need to compute $\tilde{\omega}$, which gives

$$\tilde{\omega} = \omega - df^1 \wedge df^2 = -15dz_3 \wedge dz_4 - 15dz_3 \wedge dz_6 + 15dz_4 \wedge dz_5 - 15dz_5 \wedge dz_6. \quad (39)$$

This 2-form has rank 2, and so using again (38) we obtain

$$\tilde{\omega} = d\left(\underbrace{z_4 + z_6}_{f^3}\right) \wedge d\left(\underbrace{15z_3 + 15z_5}_{f^4}\right).$$

Finally, the decomposition for ω is

$$\begin{aligned}\omega \quad = \quad & d\left(2z_2 - 3z_3 + 4z_4 - 3z_5 + 2z_6\right) \wedge d\left(2z_1 - 8z_3 + 3z_4 - 10z_5 + 3z_6\right) + \\ & + d\left(z_4 + z_6\right) \wedge d\left(15z_3 + 15z_5\right).\end{aligned}$$

The corresponding set of invariant functions is obtained by recovering the x_i-coordinates

$$y_1 = \frac{x_2^2 x_4^4 x_6^2}{x_3^3 x_5^3}, \quad y_2 = \frac{x_1^2 x_4^3 x_6^3}{x_3^8 x_5^{10}}, \quad y_3 = x_4 x_6, \quad y_4 = x_3^{15} x_5^{15}.$$

However, the reduced recurrence can be improved by considering the new set of invariant functions:

$$y_1' = \frac{y_2^{1/2} y_4^{1/3}}{y_3^{3/2}}, \quad y_2' = \frac{y_1^{1/2} y_4^{1/10}}{y_3}, \quad y_3' = y_4^{1/15}, \quad y_4' = y_3.$$

In fact the iteration map for the reduced relation is given in these variables by

$$\hat{\varphi}(y_1', y_2', y_3', y_4') = \left(y_2', y_3', y_4', \frac{y_2'^2 y_3' y_4'^2 + y_3'^4}{y_1'}\right),$$

which is (31).

5. Conclusions

We used both compatible Poisson structures and compatible pre-symplectic structures to reduce the order in recurrence relations of the form (8). In the cases we considered these approaches produced equivalent reduced relations, nevertheless we want to point out that:

(1) Compatible pre-symplectic structures always exist but compatible Poisson structures may fail to exist if B is singular.
(2) If B has rank 2 then the pre-symplectic approach will produce a reduced relation of order 2, which is the best we can expect.
(3) The Poisson approach can reduce the same recurrence to different orders (which can not happen with the pre-symplectic approach). In the example we studied (see Remark 2), the higher order reduction can be seen as an intermediate step to the final reduction, which is an interesting fact.

References

[Duist10] Duistermaat, Johannes J., *Discrete integrable systems*, Springer Monographs in Mathematics, Springer, New York, (2010).
[FoZe02] Fomin, Sergey and Zelevinsky, Andrei, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15**, no. 2, 497–529, (2002).
[FoZe02a] Fomin, Sergey and Zelevinsky, Andrei, *The Laurent phenomenon*, Adv. in Appl. Math. **28**, no. 2, 119–144, (2002).
[FoHo11] Fordy, Allan P. and Hone, Andrew, *Symplectic maps from cluster algebras*, SIGMA Symmetry Integrability Geom. Methods Appl. **7**, Paper 091, 12, (2011).
[FoHo12] Fordy, Allan P. and Hone, Andrew, *Discrete integrable systems and Poisson algebras from cluster maps*, arXiv: 1207.6072v2, (2012).
[FoMa] Fordy, Allan P. and Marsh, Robert J., *Cluster mutation-periodic quivers and associated Laurent sequences*, J. Algebraic Combin. **34**, no. 1, 19–66, (2011).
[GeShStVa] Gekhtman, M., Shapiro, M., Stolin, A. and Vainshtein, A., *Poisson structures compatible with the cluster algebra structure in Grassmannians*, Letters in Mathematical Physics, **100**, no. 2, 139–150, (2012).
[GeShVa12] Gekhtman, Michael, Shapiro, Michael and Vainshtein, Alek, *Poisson geometry of directed networks in an annulus*, J. European Math. Soc. **14**, no. 2, 541–570, (2012).
[GeShVa11] Gekhtman, Michael, Shapiro, Michael and Vainshtein, Alek, *Generalized Backlund-Darboux transformations for Coxeter-Toda flows from cluster algebra perspective*, Acta Math. **206**, no. 2, 245–310, (2011).
[GeShVa10] Gekhtman, Michael, Shapiro, Michael and Vainshtein, Alek, *Cluster Algebras and Poisson Geometry*, Mathematical Surveys and Monographs, **167**, AMS, (2010).
[GeShVa09] Gekhtman, Michael, Shapiro, Michael and Vainshtein, Alek, *Poisson geometry of directed networks in a disk* Selecta Math. **15**, no. 1, 61–103, (2009).
[GeShVa05] Gekhtman, Michael, Shapiro, Michael and Vainshtein, Alek, *Cluster algebras and Weil-Petersson forms*, Duke Math. J. **127**, no. 2, 291–311, (2005).
[GeShVa03] Gekhtman, Michael, Shapiro, Michael and Vainshtein, Alek, *Cluster algebras and Poisson geometry*, Mosc. Math. J. **3**, 899–934, (2003).

[LibMa] Libermann, Paulette and Marle, Charles-Michel, *Symplectic Geometry and Analytical Mechanics*, Mathematics and its Applications, **35**, D. Reidel Publishing Co., Dordrecht, (1987).

[QRT] Quispel, G. R. W., Roberts, J. A. G. and Thompson, C. J., *Integrable mappings and soliton equations*, Phys. Lett. A, **126**, no. 7, 419–421, (1988).