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# An almost existence theorem for non-contractible periodic orbits in cotangent bundles

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**Abstract.** Assume $M$ is a closed connected smooth manifold and $H : T^*M \to \mathbb{R}$ a smooth proper function bounded from below. Suppose the sublevel set $\{H < d\}$ contains the zero section $M$ and $\alpha$ is a non-trivial homotopy class of free loops in $M$. Then for almost every $s \in [d, \infty)$ the level set $\{H = s\}$ carries a periodic orbit $z$ of the Hamiltonian system $(T^*M, \omega_0, H)$ representing $\alpha$. Examples show that the condition $\{H < d\} \supset M$ is necessary and almost existence cannot be improved to everywhere existence.

## 1. Introduction and main result

Suppose $M$ is a smooth manifold and its cotangent bundle $\pi : T^*M \to M$ is equipped with the canonical symplectic structure $\omega_0 = -d\theta$. Here $\theta$ $(= pdq)$ denotes the canonical Liouville 1-form on $T^*M$. We view the elements of $T^*M$ as pairs $(q, p)$ where $q \in M$ and $p \in T_q^*M$. Given any function $H$ on $T^*M$, the identity $dH = \omega_0(X_H, \cdot)$ uniquely determines the Hamiltonian vector field $X_H$ on $T^*M$. The integral curves of $X_H$ are called (Hamiltonian) orbits. They preserve the level sets of the total energy $H$. Of particular interest are periodic orbits, namely orbits $\gamma : \mathbb{R} \to T^*M$ such that $\gamma(t + T) = \gamma(t)$ for some constant $T > 0$ and every $t \in \mathbb{R}$. The

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infimum[1] over such $T$ is called the period of $\gamma$. Given a family of energy levels, the question arises which levels carry a periodic orbit.

Existence of a periodic orbit on a dense set of energy levels was proved for $T^*\mathbb{R}^n$ by Hofer and Zehnder [9] in 1987 and for $T^*M$ by Hofer and Viterbo [8] in 1988. The result for $T^*\mathbb{R}^n$ was extended to existence almost everywhere by Struwe [15] in 1990. Existence of *non-contractible* periodic orbits was studied among others in 1997 by Cieliebak [2] on starshaped levels in $T^*M$, in 2000 by Gatien and Lalonde [3] employing Lagrangian submanifolds, and in 2003 by Biran, Polterovich, and Salamon [1] on $T^*M$ for $M = \mathbb{R}^n/\mathbb{Z}^n$ or $M$ closed and negatively curved. The dense existence theorem in [1] was generalized in 2006 to all closed Riemannian manifolds in [17]. Theorem A below is the corresponding almost existence theorem. In contrast the almost existence theorem of Macarini and Schlenk [13] requires finiteness of the $\pi_1$-sensitive Hofer-Zehnder capacity. An assumption that has been verified to the best of our knowledge only for such cotangent bundles which carry certain circle actions; see [11,12]. For further references concerning dense and almost existence results we refer to [5] and concerning non-contractible orbits to [7].

**Theorem A** (Almost existence)**.** *Assume $M$ is a closed connected smooth manifold and $H : T^*M \to \mathbb{R}$ is a proper[2] smooth function bounded from below. Suppose the sublevel set $\{H < d\}$ contains $M$. Then for every nontrivial homotopy class $\alpha$ of free loops in $M$ the following is true. For almost every $s \in [d, \infty)$ the level set $\{H = s\}$ carries a periodic Hamiltonian orbit $z$ that represents $\alpha$ in the sense that $[\pi \circ z] = \alpha$ where $\pi : T^*M \to M$ is the projection map.*

*Proof.* There are three main ingredients in the proof. The main player is the Biran-Polterovich-Salamon (BPS) [1] capacity $\mathrm{c}_{\mathrm{BPS}}$ whose monotonicity axiom Proposition 2.3 naturally leads to the monotone function $c_\alpha : [d, \infty) \to [0, \infty]$ defined by

$$c_\alpha(s) := \mathrm{c}_{\mathrm{BPS}}(\{H < s\}, M; \alpha). \tag{1}$$

Secondly, the existence result [17, Thm. A] concerning periodic orbits enters as follows: A priori the range of $\mathrm{c}_{\mathrm{BPS}}$ includes $\infty$ (by Definition 2.2 this is the case if no 1-periodic orbit representing $\alpha$ exists). To prove finiteness of the function $c_\alpha$ pick as an auxiliary quantity a Riemannian metric $g$ on $M$. Then using [17, Thm. A] one readily calculates that the BPS capacity of the open unit disk cotangent bundle relative to its zero section is equal to the smallest length $\ell_\alpha$ among all closed geodesics representing $\alpha$; see [17, Thm. 4.3]. The rescaling argument in Lemma 2.4 shows that

[1] Here and thoughout we use the convention $\inf \emptyset = \infty$.

[2] A map is called proper if preimages of compact sets are compact.

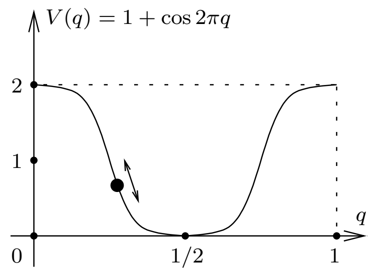


FIGURE 1. Potential energy $V$

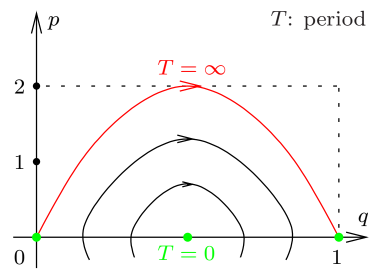


FIGURE 2. Pendulum phase portrait

the capacity of the open radius $r$ disk cotangent bundle $D_rT^*M$ is $r\ell_\alpha$. Observe that $\{H \le s\}$ is compact since $H$ is proper and bounded below. Hence the set $\{H < s\}$ is bounded and therefore contained in $D_rT^*M$ for some sufficiently large radius $r = r(s)$. Thus $c_\alpha(s) \le r(s)\ell_\alpha$ by the monotonicity axiom and this proves finiteness of $c_\alpha$.

Thirdly, by Lebesgue's last theorem, see e.g. [14], it is well known, yet amazing, that monotonicity of the map $c_\alpha : [d, \infty) \to [0, \infty)$ implies differentiability, thus Lipschitz continuity, at almost every point $s$ in the sense of measure theory. Now the key input is Theorem 3.1 whose proof is by an analogue of the Hofer-Zehnder method [10, Sec. 4.2] and which detects for each such $s$ a periodic orbit on the corresponding level set $\{H = s\}$. $\square$

**Example 1.1** (Necessary condition)**.** The condition $\{H < d\} \supset M$ cannot be dropped in Theorem A. First of all, together with $H$ being proper and bounded below, it guarantees that each level set $\{H = s\}$ is actually nonempty whenever $s \in [d, \infty)$. Now consider a pendulum. It moves on $M = S^1 = \mathbb{R}/\mathbb{Z}$ in a potential of the form $V(q) = 1 + \cos 2\pi q$; see Figure 1. The Hamiltonian $H : T^*M = S^1 \times \mathbb{R} \to \mathbb{R}$ is given by $H(q,p) = \frac{1}{2}p^2 + V(q)$; see Figure 2 for the phase portrait. Energies below the maximum value 2 of the potential $V$ do not allow for full rotations. For such low energies the pendulum can just swing hence and forth. Observe that $\{H < 1\} \not\supset M$. On the other hand, for any energy $s \in [1, 2)$ the level set $\{H = s\}$ consists of a periodic orbit which is contractible onto the stable (lower) equilibrium point $(x, y) \equiv (1/2, 0)$. So none of these orbits represents a homotopy class $\alpha \neq 0$. (For $s > 2$ the sets $\{H = s\}$ represent classes $\alpha \neq 0$. The set $\{H = 2\}$ consists of the unstable (upper) equilibrium point and two homoclinic orbits one of them indicated red in Figure 2.)

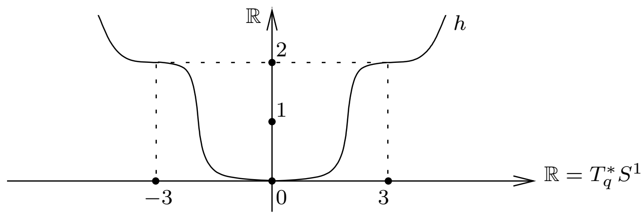


FIGURE 3. Hamiltonian $H = h(p)$ without non-constant orbits on $\{H = 2\}$

**Example 1.2** (Existence everywhere not true)**.** To see that *almost* existence in Theorem A cannot be improved to *everywhere* existence consider the case $M = S^1$ and a Hamiltonian $H : S^1 \times \mathbb{R} \to \mathbb{R}$ of the form $H(q, p) = h(p)$. More precisely, pick a proper smooth function $h \geq 0$ with $h(0) = 0$ and $h(\pm 3) = 2$ and where the points $0, \pm 3$ are the only points of slope zero; see Figure 3. Then $\{H < 1\}$ contains $M = S^1$. Moreover, the whole level set $\{H = 2\}$ consists of critical points of $H$. Therefore on $\{H = 2\}$ the Hamiltonian vector field $X_H$ vanishes identically and so all orbits are necessarily constant.

In contrast to this critical level counterexample it should be interesting to find a regular level of a smooth Hamiltonian $H$ as in Theorem A without a periodic orbit in a given homotopy class $\alpha \neq 0$. One possible way to achieve this is to start with an energy level with finitely many periodic orbits representing $\alpha$, then destroy them using the symplectic plugs constructed in [4].

For general symplectic manifolds existence may fail completely; see [18] and [16] for examples of closed symplectic manifolds admitting Hamiltonians with no non-constant periodic orbits.

## 2. Symplectic capacities

To fix notation consider $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and symplectic form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. Associate to each symplectic manifold $(N, \omega)$, of fixed dimension $2n > 0$ and possibly with boundary, a number $c(N, \omega) \in [0, \infty]$ that satisfies the axioms:

- **Monotonicity:** $c(N_1, \omega_1) \leq c(N_2, \omega)$ whenever there is a symplectic embedding $\psi : (N_1, \omega_1) \to (N_2, \omega_2)$.
- **Conformality:** $c(N, \lambda\omega) = |\lambda|\, c(N, \omega), \forall \lambda \in \mathbb{R} \setminus \{0\}$.
- **Non-Triviality:** $c(B(1), \omega_0) = c(Z(1), \omega_0) = \pi$.

Here $B(r) = \{(x, y) \in \mathbb{R}^{2n} : |x|^2 + |y|^2 < r^2\}$ is the ball of radius $r > 0$ and $Z(r) = \{(x, y) \in \mathbb{R}^{2n} : x_1^2 + y_1^2 < r^2\}$ is the symplectic cylinder of radius $r > 0$. On $(\mathbb{R}^{2n}, \omega_0)$, one checks the following re-scaling property:

$$U \subset \mathbb{R}^{2n} \text{ open} \quad \Rightarrow \quad c(\lambda U, \omega_0) = \lambda^2 c(U, \omega_0), \forall \lambda \in \mathbb{R} \setminus \{0\}.$$

A map $(N, \omega) \mapsto c(N, \omega)$ satisfying the three axioms above is called a symplectic capacity. Gromov introduced this notion in [6] and showed that

$$c_0(N, \omega) := \sup \left\{ \pi r^2 : \exists \text{ symplectic embedding } \psi : (B(r), \omega_0) \to (N, \omega) \right\}$$

is a symplectic capacity, called Gromov's width. It satisfies $c_0(N, \omega) \leq c(N, \omega)$ for any other symplectic capacity $c$. One of the consequences of the existence of a symplectic capacity is the non-squeezing theorem which asserts that

$$\exists \text{ symplectic embedding } \psi : (B(r), \omega_0) \to (Z(R), \omega_0) \quad \Leftrightarrow \quad r \leq R.$$

2.1. **Hofer-Zehnder capacity.** Hofer and Zehnder introduced in [10] a symplectic capacity defined in terms of the Hamiltonian dynamics on the underlying symplectic manifold $(N, \omega)$. Recall that a smooth function $H : N \to \mathbb{R}$ determines the Hamiltonian vector field $X_H$ by $i_{X_H}\omega = dH$. We say that a periodic orbit of $\dot{x} = X_H(x)$ is fast if its period is $< 1$. A function $H : N \to \mathbb{R}$ is called admissible if it admits a maximum and the following conditions hold:

- $0 \leq H \leq \max H < \infty$.
- $\exists K \subset N \setminus \partial N$ compact, such that $H|_{N \setminus K} = \max H$.
- $\exists U \subset N$ open and non-empty, such that $H|_U = 0$.
- $\dot{x} = X_H \circ x$ admits no non-constant fast periodic orbits.

The set of admissible Hamiltonians is denoted by $\mathcal{H}_a(N, \omega)$. Let

$$\mathrm{c}_{\mathrm{HZ}}(N, \omega) := \sup\{\max H \mid H \in \mathcal{H}_a(N, \omega)\}.$$

**Theorem 2.1** (Hofer-Zehnder)**.** $\mathrm{c}_{\mathrm{HZ}}$ *is a symplectic capacity.*

We should remark that the hard part of proving Theorem 2.1 is to show that $\mathrm{c}_{\mathrm{HZ}}$ satisfies the non-triviality axiom.

2.2. **BPS relative capacity.** Fix a closed manifold $M$. The components $\mathcal{L}_\alpha M$ of the free loop space $\mathcal{L}M := C^\infty(S^1, M)$ are labelled by the elements $\alpha = [\gamma]$ of the set $\tilde{\pi}_1(M)$ of homotopy classes of free loops $\gamma$ in $M$. Here and throughout we identify $S^1$ with $\mathbb{R}/\mathbb{Z}$ and think of $\gamma$ as a smooth map $\gamma : \mathbb{R} \to M$ that satisfies $\gamma(t + 1) = \gamma(t)$ for every $t \in \mathbb{R}$. A function $H \in C_0^\infty(S^1 \times T^*M)$ determines a 1-periodic family of compactly supported vector fields $X_{H_t}$ on $T^*M$ by $dH_t = \omega_0(X_{H_t}, \cdot)$. Let

$$\mathcal{P}_1(H; \alpha) := \{z : S^1 \to T^*M \mid \dot{z}(t) = X_{H_t}(z(t)) \, \forall t \in S^1, [\pi \circ z] = \alpha\}$$

be the set of 1-periodic orbits of $X_{H_t}$ whose projections to $M$ represent $\alpha$.

**Definition 2.2.** Following [1] assume $W \subset T^*M$ is an open subset which contains the zero section $M$. For any constant $b > 0$ consider the set

$$\mathcal{H}_b(W) := \left\{ H \in C_0^\infty(S^1 \times W) \,\middle|\, m_0(H) := \max_{S^1 \times M} H \leq -b \right\}.$$

The BPS capacity of $W$ relative $M$ and with respect to $\alpha \in \tilde{\pi}_1(M)$ is defined by

$$\mathrm{c}_{\mathrm{BPS}}(W, M; \alpha) := \inf \{ b > 0 \mid \mathcal{P}_1(H; \alpha) \neq \emptyset \text{ for every } H \in \mathcal{H}_b(W) \}. \quad (2)$$

Note that $\mathrm{c}_{\mathrm{BPS}}$ takes values in $[0, \infty]$ since we use the convention $\inf \emptyset = \infty$. Furthermore, the BPS capacity is a relative symplectic capacity.

**Proposition 2.3** (Monotonicity [1, Prop. 3.3.1])**.** *If $W_1 \subset W_2 \subset T^*M$ are open subsets containing $M$ and $\alpha \in \tilde{\pi}_1(M)$, then $\mathrm{c}_{\mathrm{BPS}}(W_1, M; \alpha) \leq \mathrm{c}_{\mathrm{BPS}}(W_2, M; \alpha)$.*

Fix a Riemannian metric on $M$ and constants $r, b > 0$. Denote by $DT^*M$ the open unit disk cotangent bundle and by $D_rT^*M$ the one of radius $r$. Observe that

$$H \in \mathcal{H}_b(DT^*M) \quad \Longleftrightarrow \quad H_r \in \mathcal{H}_{rb}(D_rT^*M) \quad (3)$$

whenever the Hamiltonians $H$ and $H_r$ are related by $H_r(t, q, p) = r \cdot H(t, q, \frac{p}{r})$. In addition, pick $\alpha \in \tilde{\pi}_1(M)$. Then there is the crucial bijection

$$\mathcal{P}_1(H; \alpha) \to \mathcal{P}_1(H_r; \alpha) : (x, y) \mapsto (x, ry) \quad (4)$$

asserting that the 1-periodic orbits of $H$ correspond naturally with those of $H_r$.

**Lemma 2.4** (Rescaling)**.** $\mathrm{c}_{\mathrm{BPS}}(D_rT^*M, M; \alpha) = r \cdot \mathrm{c}_{\mathrm{BPS}}(DT^*M, M; \alpha)$.

*Proof.* By definition (2) of the BPS capacity we obtain that

$$\begin{aligned}
&\mathrm{c}_{\mathrm{BPS}}(D_rT^*M, M; \alpha) \\
&= \inf \{ rb > 0 \mid \mathcal{P}_1(H_r; \alpha) \neq \emptyset \text{ for every } H_r \in \mathcal{H}_{rb}(D_rT^*M) \} \\
&= r \cdot \inf \{ b > 0 \mid \mathcal{P}_1(H; \alpha) \neq \emptyset \text{ for every } H \in \mathcal{H}_b(DT^*M) \} \\
&= r \cdot \mathrm{c}_{\mathrm{BPS}}(DT^*M, M; \alpha)
\end{aligned}$$

where the second step uses (3) and (4). □

**Corollary 2.5.** $\mathrm{c}_{\mathrm{BPS}}(D_rT^*M, M; \alpha) = r\ell_\alpha$ *where $\ell_\alpha$ is the smallest length among all closed curves representing $\alpha$.*

*Proof.* $\mathrm{c}_{\mathrm{BPS}}(DT^*M, M; \alpha) = \ell_\alpha$ by [17, Thm. 4.3]. Apply Lemma 2.4. □

## 3. The Hofer-Zehnder method

Assume the Hamiltonian $H : T^*M \to \mathbb{R}$ is smooth, proper, and bounded from below and a sublevel set $\{H < d\}$ contains $M$. Fix a non-trivial homotopy class $\alpha$ of free loops in $M$. Consider the monotone function $c_\alpha$ defined on the interval $[d, \infty)$ by (1). By Lebesgue's last theorem, see e.g. [14, p. 401], the function $c_\alpha$ is differentiable at almost every point in the sense of measure theory.

**Theorem 3.1.** *Assume $s_0 \in [d, \infty)$ is a regular value of $H$ and $c_\alpha$ is Lipschitz continuous at $s_0$. Then the hypersurface $S = H^{-1}(s_0)$ carries a periodic orbit $z_T$ of $X_H$ that represents $\alpha$ and where $T > 0$ is the period.*

*Proof.* The proof is an adaption of the Hofer-Zehnder method [10, Sec. 4.2] to the case at hand. To emphasize this we mainly keep their notation. Fix $s_0$ as in the hypothesis of the theorem. Then $S_0 := H^{-1}(s_0)$ is a hypersurface[3] in $T^*M$ by the inverse function theorem. It is compact since $H$ is proper and it bounds the open set $\dot{B}_0 := \{H < s_0\}$ since $H$ is bounded below. Furthermore, by the implicit function theorem and compactness of $S_0$ there is a constant $\mu > 0$ such that $s_0 + \varepsilon$ is a regular value of $H$ and $S_\varepsilon := H^{-1}(s_0 + \varepsilon)$ is diffeomorphic to $S_0$ whenever $\varepsilon \in [-\mu, \mu]$. Note that $S_\varepsilon$ bounds the open set $\dot{B}_\varepsilon := \{H < s_0 + \varepsilon\}$ which itself contains the zero section $M$ of $T^*M$. Furthermore, since $c_\alpha$ is Lipschitz continuous at $s_0$ there is a constant $L > 0$ such that

$$c(\varepsilon) - c(0) \leq L\varepsilon, \qquad c(\varepsilon) := c_\alpha(s_0 + \varepsilon), \tag{5}$$

for every $\varepsilon \in [-\mu, \mu]$; otherwise, choose $\mu > 0$ smaller. We proceed in three steps I–III.

I. Pick $\tau \in (0, \mu)$. Then there is a Hamiltonian $K \in C_0^\infty(S^1 \times \dot{B}_0)$ whose maximum over the zero section satisfies

$$-c(0) < m_0(K) \leq -\left(c(0) - L\tau\right)$$

and which does not admit any 1-periodic orbit representing $\alpha$. Indeed if no such $K$ exists, then $c_{\mathrm{BPS}}(\dot{B}_0, M; \alpha) \leq c(0) - L\tau$ by definition (2) of the BPS capacity. But $c(0) = c_\alpha(s_0) = c_{\mathrm{BPS}}(\dot{B}_0, M; \alpha)$ and we obtain the contradiction $c(0) \leq c(0) - L\tau$. Now pick a smooth function $f : \mathbb{R} \to [-3L\tau, 0]$ such that

$$\begin{aligned} f(s) &= -3L\tau && \text{if} \quad s \leq 0 \\ f(s) &= 0 && \text{if} \quad s \geq \tfrac{\tau}{2} \\ 0 < f'(s) &\leq 7L && \text{if} \quad 0 < s < \tfrac{\tau}{2} \end{aligned}$$

[3]A hypersurface is a smooth submanifold of codimension 1.

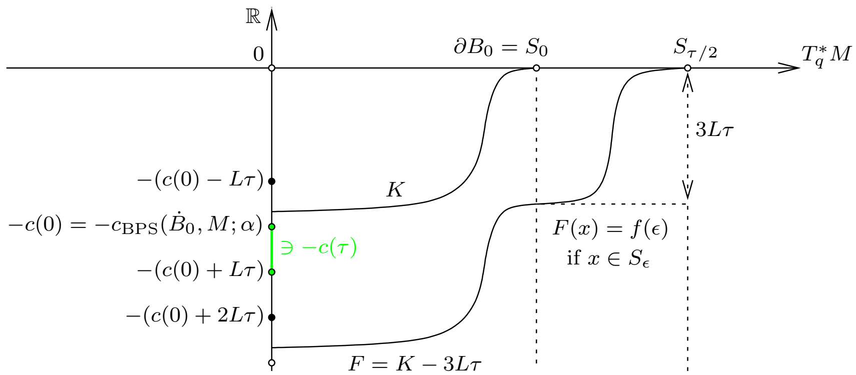


FIGURE 4. Hamiltonians $F \in \mathcal{H}_{c(\tau)}(\dot{B}_\tau)$ and $K$ with $\mathcal{P}_1(K;\alpha) = \emptyset$

and consider the Hamiltonian $F \in C_0^\infty(S^1 \times \dot{B}_\tau)$ defined by

$$\begin{array}{lll} F(x) = K(x) - 3L\tau & \text{if} & x \in \dot{B}_0 \\ F(x) = f(\varepsilon) & \text{if} & x \in S_\varepsilon = H^{-1}(s_0 + \varepsilon),\ 0 \le \varepsilon < \tau \\ F(x) = 0 & \text{if} & x \notin \dot{B}_\tau \end{array}$$

and illustrated by Figure 4. By (5) the Hamiltonian $F$ satisfies the estimate

$$m_0(F) = m_0(K) - 3L\tau \le -(c(0) - L\tau) - 3L\tau < -(c(0) + L\tau) \le -c(\tau).$$

Since $m_0(F) \le -c_{\mathrm{BPS}}(\dot{B}_\tau, M; \alpha)$ the definition (2) of the BPS capacity shows that the set $\mathcal{P}_1(F;\alpha)$ is not empty. In other words, there is a 1-periodic orbit $z$ of $X_F$ that represents $\alpha$. Observe that $z$ cannot intersect $\dot{B}_0$: Due to compact support the open set $\dot{B}_0$ is invariant under the flow of $K$. But the flows of $K$ and $K - 3L\tau = F$ coincide. Thus, if $z$ intersects $\dot{B}_0$, then it stays completely inside. But this is impossible since $\mathcal{P}_1(K;\alpha) = \emptyset$. On the other hand, since $\alpha \neq 0$ the orbit $z$ of $X_F$ is non-constant and therefore it must intersect the regions foliated by the hypersurfaces $S_\varepsilon$ where $0 < \varepsilon < \frac{\tau}{2}$. But each of them is a level set of $F$, hence invariant under the flow of $X_F$. This shows that $z$ lies on $S_\varepsilon$ for some $0 < \varepsilon < \frac{\tau}{2}$.

II. Repeat the argument for each element of a sequence $\tau_j \to 0$ to obtain sequences $F_j$ and $\varepsilon_j$ and a sequence $z_j$ of 1-periodic orbits of $X_{F_j}$ that lie on $S_{\varepsilon_j}$ and where $\varepsilon_j \to 0$. Next we interpret each $z_j$ as a $T_j$-periodic orbit of $X_H$ by rescaling time. Most importantly, the periods $T_j$ are uniformly bounded from above by $7L$. To see this note that on the open set

$$U := \bigcup_{\varepsilon\in(-\mu,\mu)} S_\varepsilon = \bigcup_{\varepsilon\in(-\mu,\mu)} H^{-1}(s_0+\varepsilon)$$

the Hamiltonian $H$ is obviously given by $H(x) = s_0 + \varepsilon$ whenever $x \in S_\varepsilon$. For each $\tau_j$ and each $\varepsilon \in [0, \tau_j)$ we have

$$F_j(x) = f_j(H(x) - s_0) = f_j(\varepsilon)$$

for every $x \in S_\varepsilon$. At such $x$ use the definition of $X_{F_j}$ and the chain rule to get

$$\omega_0\left(X_{F_j}, \cdot\right) = dF_j = f_j'(H - s_0)dH = \omega_0\left(f_j'(\varepsilon)X_H, \cdot\right).$$

Thus, because $z_j$ lies on $S_{\varepsilon_j}$, it satisfies the equation

$$\dot z_j(t) = X_{F_j} \circ z_j(t) = T_j \cdot X_H \circ z_j(t), \qquad T_j := f_j'(\varepsilon_j),$$

and the periodic boundary condition $z_j(t+1) = z_j(t)$ for every $t \in \mathbb{R}$.

III. Uniform boundedness of the periods $T_j$ is crucial in the following proof of existence of a 1-periodic orbit $z$ of $X_H$ which lies on the original level hypersurface $S_0 = H^{-1}(s_0)$ and represents the given class $\alpha$. Indeed note that $S_{\varepsilon_j} \subset \{H \le s_0 + \mu\} =: B_\mu$ and that $B_\mu$ is compact since $H$ is proper and bounded below. In other words, the sequence of loops $z_j$ is uniformly bounded in $C^0$. Concerning $C^1$ we obtain the uniform estimate

$$|\dot z_j(t)| = |T_j| \cdot |X_H \circ z_j(t)| \le 7L \left\| X_H \right\|_{C^0(B_\mu)}$$

for all $t \in S^1$ and $j \in \mathbb{N}$. Therefore by the Arzelà-Ascoli theorem there is a subsequence, still denoted by $z_j$, which converges in $C^0$ and by using the equation for $z_j$ even in $C^\infty$ to a smooth 1-periodic solution $z$ of the equation $\dot z = T \cdot X_H(z)$ where $T = \lim_{j\to\infty} T_j$. Since $\varepsilon_j \to 0$ the orbit $z$ takes values on the desired level hypersurface $S_0 = H^{-1}(s_0)$. To prove that $z = (x, y)$ represents the same class $\alpha$ as does each $z_j = (x_j, y_j)$ we need to show that $[x] = [x_j]$ for some $j$. To see this consider the injectivity radius $\iota > 0$ of the compact Riemannian manifold $(M, g)$ and pick $j$ sufficiently large such that the Riemannian distance between $x(t)$ and $x_j(t)$ is less than $\iota/2$ for every $t \in S^1$. Setting $\exp_{x(t)} \xi(t) = x_j(t)$ provides the desired homotopy $h_\lambda(t) = \exp_{x(t)} \lambda\xi(t)$ between $h_0 = x$ and $h_1 = x_j$.
Reparametrize time to obtain the $T$-periodic solution $z_T(t) := z(t/T)$ of

$$\dot z_T(t) = \tfrac{1}{T}\dot z(t/T) = X_H \circ z(t/T) = X_H \circ z_T(t)$$

which obviously represents the same class $\alpha$ as $z$. Since $\alpha \ne 0$ the loop $z_T$ cannot be constant and so the period necessarily satisfies $T > 0$. This concludes the proof of Theorem 3.1. $\square$

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