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Dynkin diagrams and spectra of graphs

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1. Introduction

Dynkin diagrams first appeared in [20] in the connection with classification of simple Lie groups. Among Dynkin diagrams a special role is played by the simply laced Dynkin diagrams A_n, D_n, E_6, E_7 and E_8. Dynkin diagrams are closely related to Coxeter graphs that appeared in geometry (see [8]). After that Dynkin diagrams appeared in many braches of mathematics and beyond, em particular em representation theory.

In [22] P. Gabriel introduced a notion of a quiver (directed graph) and its representations. He proved the famous Gabriel's theorem on representations of quivers over algebraically closed field.

Let Q be a finite quiver and $\bar{Q}$ the undirected graph obtained from Q by deleting the orientation of all arrows.

Theorem 1.1. (Gabriel's Theorem). *A connected quiver Q is of finite type if and only if the graph $\bar{Q}$ is one of the following simply laced Dynkin diagrams: A_n, D_n, E_6, E_7 or E_8.*

I.N. Bernstein, I.M. Gelfand and V.A. Ponomarev [5] gave a proof of Gabriel's Theorem using roots, Weyl groups and Coxeter functors.

The terms "tame type" and "wild type" were introduced by P. Donovan and M.R. Freislich [16]. Extended Dynkin diagrams or Euclidean diagrams are $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ (see, for example, [2]). Tame quivers in terms of extended Dynkin diagrams were classified by L.A. Nazarova [39] and by P. Donovan–M.R. Freislich [16]. For finite dimensional algebras and some other algebraic structures the tame-wild dichotomy problem was solved by Yu.A. Drozd [17]–[19]. The theory of K-species was first considered by P. Gabriel in [23]. He obtained the characterization of K-species of finite type in a special case. His result was extended by V. Dlab and C.M. Ringel (see [14, Theorem B]).

Theorem 1.2. (Theorem B). *A K-species is of a finite type if and only if its diagram is a finite disjoint union of Dynkin diagrams.*

The problem of the ubiquity of the symply laced Dynkin diagrams A_n, D_n, E_n was formulated by V.I. Arnold [1] as follows.

A-D-E classification. The Coxeter-Dynkin graphs A_n, D_n and E_n appear in many independent classification theorems. For instance

(a) the classification of the platonic solids (or finite orthogonal groups in euclidean 3-space),

(b) the classification of the categories of linear spaces and maps (representations of quivers,

(c) the classification of the singularities of algebraic hypersurfaces, with a definite intersection form of the neighboring smooth fibre,

(d) the classification of the critical points of functions having no moduli,

(e) the classification of the Coxeter groups generated by reflections, or, of Weyl groups with roots of equal length.

The problem is to find the common origin of all A-D-E classification theorems and to substitute a priori proofs to a posteriori verifications of the parallelism of the classifications. An introduction to the A-D-E-problem can be found in [30].

Dynkin diagrams and extended Dynkin diagrams are widely used in the study of generalized Cartan matrices and Kac–Moody algebras [2]–[4], [6], [31], [35], [36], [40] and [42].

Let G be a finite graph without loops and multiple edges (G is a finite simple graph). J.H. Smith [41] formulated the following result:

Theorem 1.3. *Let G be a finite simple graph with the spectral radius (index) r_G. Then $r_G = 2$ if and only if each connected component of G is one of the extended Dynkin diagram $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$. Moreover, $r_G < 2$ if and only if each connected component of G is one of Dynkin diagrams A_n, D_n, E_6, E_7, E_8.*

For the full proofs of this Smith's theorem see, for example, [27, chapter I and Appendix I], [37] and [21, Theorem 2.12]. Note that Theorem 1.3 was obtained also in [33, Theorem 5.1] and [7]. In 1975 (see [11]) D.M. Cvetkovich and I. Gutman introduced for extended Dynkin diagrams of type $\tilde{A}$ and $\tilde{D}$ the symbols C_n and W_n. Moreover, they used the following notations: P_n for A_n; Z_n for D_{n+2}, T_1 for E_6, T_2 for E_7, T_3 for E_8, T_4 for $\tilde{E}_6$, T_5 for $\tilde{E}_7$ and T_6 for $\tilde{E}_8$.

The following terminology is used in [12, pp. 77-79]: "Smith's graphs" means extended Dynkin diagrams and "reduced Smith's graphs" means simply laced Dynkin diagrams A_n, D_n, E_6, E_7, E_8 (see also [9] and [10]).

In this paper we consider spectral properties of graphs based on Perron-Frobenius theory of non-negative matrices. We will use terminology and results from [29, Section 6.5] and [25].

2. Symmetric non-negative matrices

Let G be an undirected finite graph without loops and multiple edges, i.e., G is a **finite simple graph**.

Let $VG = \{1, \dots, n\}$ be the vertex set of G and EG be the edge set of G. Two vertices i and j are called adjacent if they are connected by an edge.

The adjacency matrix $[G]$ of a simple graph with n vertices is a square matrix $[G] = (\alpha_j)$ of order n, whose (i,j)-entry α_{ij} is 1, if the vertices i and j are adjacent, otherwise $\alpha_{ij} = 0$. Therefore, $[G]$ is a symmetric $(0,1)$-matrix with zero main diagonal.

Denote by $M_n(\mathbb{R})$ the ring of all $n \times n$ matrices with real entries. Let $A = (a_{ij}) \in M_n(\mathbb{R})$ be a non-negative symmetric permutationally irreducible matrix.

From the Perron-Frobenius Theorem it follows that A has the largest positive eigenvalue r_A such that any eigenvalue λ of A one has that $|\lambda| \leq r_A$, and there exists a positive eigenvector $\vec{z} = (z_1, \dots, z_n)^T$ with $A\vec{z} = r_A\vec{z}$. We give the next.

Theorem 2.1. *Let $A = (a_{ij}) \in M_n(\mathbb{R})$ be a nonnegative symmetric permutationally irreducible matrix and B be its proper main submatrix. Then $r_B < r_A$.*

Before the proof of the theorem we give necessary information about the properties of A.

Lemma 2.1. [26] *Eigenvectors of a matrix belonging to different eigenvalues are orthogonal.*

Corollary 2.1. *Let $A \in M_n(\mathbb{R})$ be a permutationally irreducible symmetric matrix and $\vec{z} = (z_1, \ldots, z_n)^T$ be its positive eigenvector, then $A\vec{z} = r_A\vec{z}$.*

Proof. Suppose that $A\vec{z} = \lambda\vec{z}$ and $\lambda \neq r_A$. Let $\vec{w} = (w_1, \ldots, w_n)^T$ be a positive eigenvector of A with eigenvalue r_A. Then by Lemma 2.1 the inner product $(\vec{z}, \vec{w})$ is zero. We obtain a contradiction:

$$\sum_{i=1}^{n} z_i w_i > 0.$$

□

Now we give a proof of Theorem 2.1.

Proof. Let B be a proper principal $m \times m$-submatrix of A. We enumerate the rows and columns of A such that:

$$A = \left(\begin{array}{ccc|c} B_1 & \ldots & 0 & X_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & B_t & X_t \\ \hline X_1^T & \ldots & X_t^T & \mathrm{C} \end{array}\right),$$

where $B = \begin{pmatrix} B_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & B_t \end{pmatrix}$ and the matrices $B_1, \ldots, B_t$ are permutationally irreducible.

We may assume that $r_B = r_{B_1}$, $B_1 \in M_{m_1}(\mathbb{R}), \ldots, B_t \in M_{m_t}(\mathbb{R})$, $m_1 + \ldots + m_t = m$. Then, $C \in M_{n-m}(\mathbb{R})$ and $X = \begin{pmatrix} X_1 \\ \vdots \\ X_t \end{pmatrix}$, where $X_i \in M_{m_i \times (n-m)}(\mathbb{R})$.

The matrix A is permutationally irreducible, so $X_1 \neq 0$.

Let $\vec{z} = (z_1, \dots, z_n)^T$ be the Perron-Frobenius positive eigenvector of A, i.e., $A\vec{z} = r_A\vec{z}$. Denote by $\vec{z}_s = (z_1, \dots, z_{m_1})$ th evector formed by the first m_1 coordinates of $\vec{z}$ and by $\vec{z}_e = (z_{n-m+1}, \dots, z_n)$.

Then we obtain: $B_1\vec{z}_s + X_1\vec{z}_e = r_A\vec{z}_s$. Obviously the non-negative vector $X_1\vec{z}_e$ is nonzero (vector $\vec{z}_e$ is positive and $X_1 \neq 0$ and non-negative). We have $y_i \geq 0$ for $i = 1, \dots, m_1$. Therefore $y_i \leq r_A z_i$ for $i = 1, \dots, m_1$ and there exists $1 \leq k \leq m_1$ such that $y_k < r_A z_k$. Let $\vec{f} = (f_1, \dots, f_{m_1})^T$ be a Perron-Frobenius vector of B_1, so $B_1\vec{f} = r_B\vec{f}$. Then $(\vec{z}_s, B_1\vec{f}) = (\vec{z}_s, r_{B_1}\vec{f}) = r_{B_1}(\vec{z}_s, \vec{f}) = (B_1\vec{z}_s, \vec{f}) < (r_A\vec{z}_s, \vec{f}) = r_A(\vec{z}_s, \vec{f})$, i.e., $r_{B_1}(\vec{z}_s, \vec{f}) < r_A(\vec{z}_s, \vec{f})$. Then $\vec{z}_s, \vec{f}) > 0$ as inner product of positive vectors. Therefore $r_B = r_{B_1} < r_A$. Theorem is proved. □

3. Spectra of Dynkin diagrams and extended Dynkin diagrams

In this section we give a list of characteristic polynomials and spectra of Dynkin diagrams and of extended Dynkin diagrams.

Theorem 3.1. (L. Kronecker, [32]) *Suppose that all the real roots of a monic polynomial with integer coefficients belong to the interval* $[-2, 2]$ *and are given in the form*

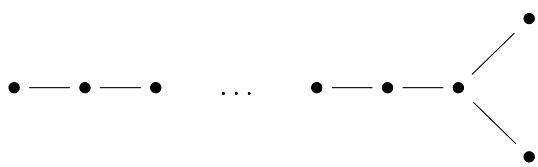
$$2\cos\alpha, 2\cos\beta, 2\cos\gamma, \dots.$$

Then the angles α, β, γ, … *are rational multiples of* $\pi/2$.

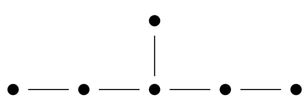
The following simple graphs are simply laced Dynkin diagrams:

A_n, $n \geq 1$:

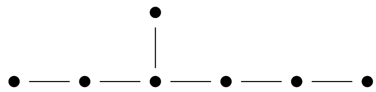
D_n, $n \geq 4$:



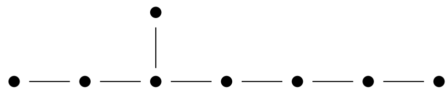
E_6 :



E_7 :

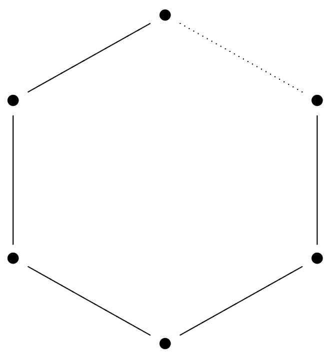


E_8 :

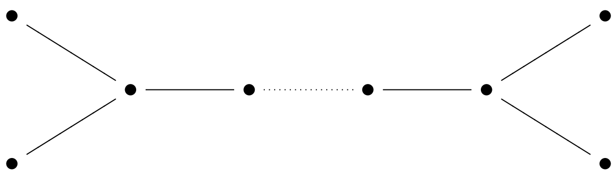


The following simple graphs are extended versions of simply laced Dynkin diagrams:

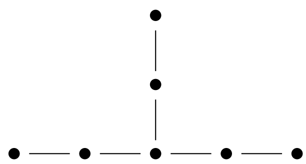
$\tilde{A}_n \, (n \geq 2)$:



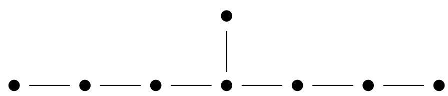
$\tilde{D}_n \ (n \geq 4)$:



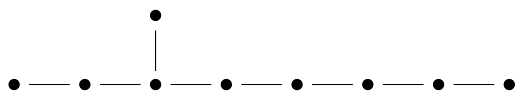
$\tilde{E}_6$:



$\tilde{E}_7$:



$\tilde{E}_8$:



Often extended Dynkin diagrams are called Euclidean diagrams.

Proposition 3.1. *For the Dynkin diagram A_n $(n \geq 1)$ we have*

$$\chi_{A_n}(x) = \prod_{1 \leq k \leq n} \left(x - 2\cos\frac{k\pi}{n+1} \right)$$

Consequently,

$$S(A_n) = \left\{ 2\cos\frac{k\pi}{n+1} \mid k = 1, \ldots, n \right\}$$

and $r_{A_n} = 2\cos\frac{\pi}{n+1}$, where $S(A_n)$ denotes the spectrum of A_n.

Proposition 3.2. *For the Dynkin diagram D_n $(n \geq 4)$ we have*

$$\chi_{D_n}(x) = x \left(\prod_{0 \leq k \leq n-2} (x - 2\cos\frac{(1+2k)\pi}{2(k-1)} \right).$$

Consequently, $S(D_n)$ consists of zero and of the following set:

$$\left\{ 2\cos\frac{(1+2k)\pi}{2(n-1)} \mid k = 0, \ldots, n-2 \right\}$$

and $r_{D_n} = 2\cos\frac{\pi}{2(n-1)}$.

Proposition 3.3. *For the Dynkin diagram E_6 we have*

$$\chi_{E_6}(x) = x^6 - 5x^4 + 5x^2 - 1 = \prod_{1 \leq k \leq 6} \left(x - 2\cos\frac{m_k\pi}{12} \right),$$

where $m_k = 1, 4, 5, 7, 8, 11$. Then

$$S(E_6) = \left\{ 2\cos\frac{m_k\pi}{12} \mid m_k = 1, 4, 5, 7, 8, 11 \right\}$$

and $r_{E_6} = 2\cos\frac{\pi}{12}$.

Proposition 3.4. *For the Dynkin diagram E_7 we have*

$$\chi_{E_7}(x) = x(x^6 - 6x^4 + 9x^2 - 3) = \prod_{1 \leq k \leq 7} \left(x - 2\cos\frac{m_k\pi}{18} \right),$$

where $m_k = 1, 5, 7, 9, 11, 13, 17$. *Then*

$$S(E_7) = \left\{2\cos\frac{m_k\pi}{18} \mid m_k = 1, 5, 7, 9, 11, 13, 17\right\}$$

and $r_{E_7} = 2\cos\frac{\pi}{18}$.

Proposition 3.5. *For the Dynkin diagram* E_8 *we have*

$$\chi_{E_8}(x) = x^8 - 7x^6 + 14x^4 - 8x^2 + 1 = \prod_{1\le k\le 8}\left(x - 2\cos\frac{m_k\pi}{30}\right),$$

where $m_k = 1, 7, 11, 13, 17, 19, 23, 29$. *Then*

$$S(E_8) = \left\{2\cos\frac{m_k\pi}{30} \mid m_k = 1, 7, 11, 13, 17, 19, 23, 29\right\}$$

and $r_{E_8} = 2\cos\frac{\pi}{30}$.

Proposition 3.6. *For the extended Dynkin diagram* $\tilde{A}_n$ $(n \ge 2)$ *we have*

$$\chi_{\tilde{A}_n}(x) = \mu^{n+1} + \mu^{-n-1} - 2 = \prod_{1\le k\le n}\left(x - 2\cos\frac{2k\pi}{n+1}\right),$$

where $x = \mu + \frac{1}{\mu}$. *Then consequently,*

$$S(\tilde{A}_n) = \left\{2\cos\frac{2k\pi}{n+1} \mid k = 0, \dots, n\right\}$$

and $r_{\tilde{A}_n} = 2$.

Proposition 3.7. *For the extended Dynkin diagram* $\tilde{D}_n$ $(n \ge 4)$ *we have*

$$\chi_{\tilde{D}_n}(x) = \chi_{\tilde{A}_3}(x)\chi_{n-3}(x) = x^2(x^2 - 4)\prod_{0\le k\le n-3}\left(x - 2\cos\frac{k\pi}{n-2}\right).$$

Then

$$S(\tilde{D}_n) = \left\{2\cos\frac{k\pi}{n-2} \mid k = 1, \dots, n-3\right\} \cup [-2, 0, 0, 2]$$

and $r_{\tilde{D}_n} = 2$.

Proposition 3.8. *For the extended Dynkin diagrams $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ we have*

$$\chi_{\tilde{E}_6}(x) = x(x^2-1)^2(x^2-4);$$

$$\chi_{\tilde{E}_7}(x) = x(x^2-1)(x^2-4)\prod_{1\le k\le 3}(x - 2\cos\tfrac{k\pi}{4});$$

$$\chi_{\tilde{E}_8}(x) = x(x^2-1)(x^2-4)\prod_{1\le k\le 4}(x - 2\cos\tfrac{k\pi}{5}).$$

Then

$$S(\tilde{E}_6) = [0, \pm 1, \pm 1, \pm 2] \text{ and } r_{\tilde{E}_6} = 2.$$

$$S(\tilde{E}_7) = \left\{2\cos\tfrac{k\pi}{4} \mid k = 1,2,3\right\} \cup \{0, \pm 1, \pm 2\} \text{ and } r_{\tilde{E}_7} = 2.$$

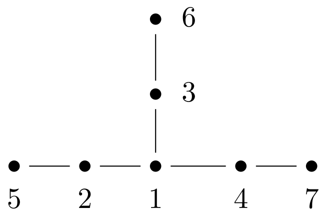
$$S(\tilde{E}_8) = \left\{2\cos\tfrac{k\pi}{5} \mid k = 1,2,3,4\right\} \cup \{0, \pm 1, \pm 2\} \text{ and } r_{\tilde{E}_8} = 2.$$

4. **Perron-Frobenius vectors of extended Dynkin diagrams**

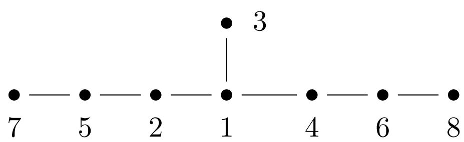
We consider simply laced extended Dynkin diagrams and its Perron-Frobenius vectors.

We give the list of these graphs with the numbering of vertices suitable for us:

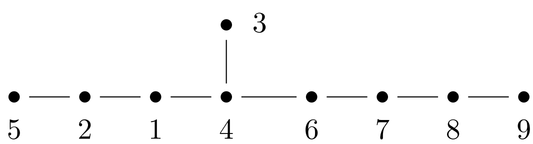
$\tilde{E}_6$:



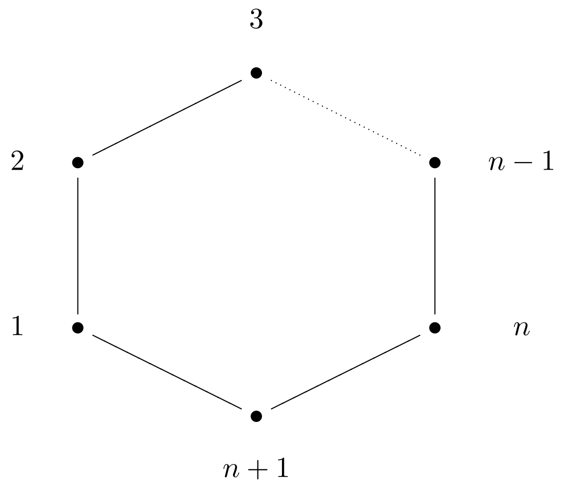
$\tilde{E}_7$:



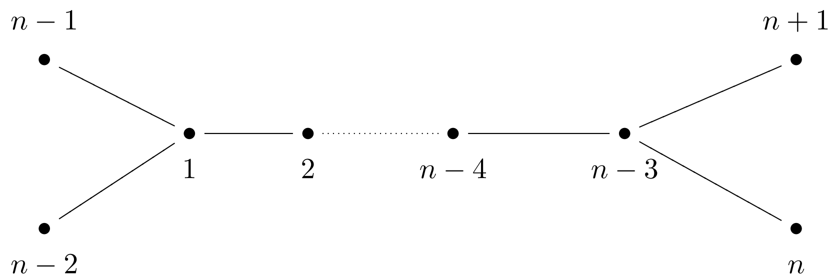
$\tilde{E}_8$:



$\tilde{A}_n\ (n \geq 2)$:



$\tilde{D}_n$ $(n \geq 4)$:



Case $\tilde{E}_6$.
The adjacency matrix is

$$[\tilde{E}_6] = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Let $\vec{z} = (z_1, z_2, z_3, z_4, z_5, z_6, z_7)^T$ be a positive eigenvector of $\tilde{E}_6$.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{bmatrix} = \begin{bmatrix} \lambda z_1 \\ \lambda z_2 \\ \lambda z_3 \\ \lambda z_4 \\ \lambda z_5 \\ \lambda z_6 \\ \lambda z_7 \end{bmatrix}$$

$$\begin{cases} z_2 + z_3 + z_4 = \lambda z_1 \\ z_1 + z_5 = \lambda z_2 \\ z_1 + z_6 = \lambda z_3 \\ z_1 + z_7 = \lambda z_4 \\ z_2 = \lambda z_5 \\ z_3 = \lambda z_6 \\ z_4 = \lambda z_7 \end{cases}$$

$z_2 + z_3 + z_4 = \lambda(z_5 + z_6 + z_7) = \lambda z_1$;
$z_1 = z_5 + z_6 + z_7$, $4z_1 = \lambda^2 z_1$, i.e., $\lambda = 2$.
$z_2 = 2z_5$, $z_3 = 2z_6$ and $z_2 = z_3 = z_4 = 2$, $2z_1 = 6$, $z_1 = 3$.
We obtain $\vec{z} = (3, 2, 2, 2, 1, 1, 1)^T$.

Case $\tilde{E}_7$.
The adjacency matrix is

$$[\tilde{E}_7] = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

As above

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \end{bmatrix} = \lambda \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \end{bmatrix}$$

$$\begin{cases} z_1 + z_3 + z_4 = \lambda z_1 \\ z_1 + z_5 = \lambda z_2 \\ z_1 = \lambda z_3 \\ z_1 + z_6 = \lambda z_4 \\ z_2 + z_7 = \lambda z_5 \\ z_4 + z_8 = \lambda z_6 \\ z_5 = \lambda z_7 \\ z_6 = \lambda z_8 \end{cases}$$

We have $z_6 = \lambda z_8$, $z_4 = (\lambda^2 - 1)z_8$, $z_1 = (\lambda^3 - 2\lambda)z_8$,
$z_3 = (\lambda^2 - 2)z_8$, $z_2 = (\lambda^4 - 4\lambda^2 + 3)z_8$,
$z_5 = (\lambda^5 - 5\lambda^3 + 5\lambda)z_8$, $z_7 = (\lambda^4 - 5\lambda^2 + 5)z_8$.
Let $z_8 = 1$. Then from $z_2 + z_7 = \lambda z_5$ it follows that
$\lambda^4 - 4\lambda^2 + 3 + \lambda^4 - 5\lambda^2 + 5 = \lambda^6 - 5\lambda^4 + 5\lambda^2$, i.e.,
$\lambda^6 - 7\lambda^4 + 14\lambda^2 - 8 = 0$.

Obviously, $2^6 - 7 \cdot 2^4 + 14 \cdot 2^2 - 2^3 = 2^2(16 - 28 + 14 - 2) = 0$, i.e., $\lambda = 2$ is a root. Therefore $\vec{z} = (4, 3, 2, 3, 2, 2, 1, 1)^T$.

Case $\tilde{E}_8$.

The adjacency matrix is

$$[\tilde{E}_8] = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

As above

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \end{bmatrix} = \lambda \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \end{bmatrix}$$

$$\begin{cases} z_2 + z_3 + z_4 = \lambda z_1 \\ z_1 + z_5 = \lambda z_2 \\ z_1 = \lambda z_3 \\ z_1 + z_6 = \lambda z_4 \\ z_2 = \lambda z_5 \\ z_4 + z_7 = \lambda z_6 \\ z_6 + z_8 = \lambda z_7 \\ z_7 + z_9 = \lambda z_8 \\ z_8 = \lambda z_9 \end{cases}$$

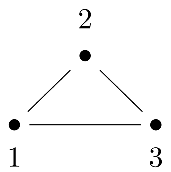
We have $z_8 = \lambda z_9$, $z_7 = (\lambda^2 - 1)z_9$, $z_6 = (\lambda^3 - 2\lambda)z_9$,
$z_4 = (\lambda^4 - -3\lambda^2 + 1)z_9$, $z_1 = (\lambda^5 - 4\lambda^3 + 3\lambda)z_9$,
$z_3 = (\lambda^4 - 4\lambda^2 + 3)z_9$, $z_2 = (\lambda^6 - 6\lambda^4 + 10\lambda^2 - 4)z_9$,
$z_5 = 9\lambda^7 - 7\lambda^5 = 14\lambda^3 - 7\lambda)z_9$.

Let $z_9 = 1$. From $z_2 = \lambda z_5$ we obtain

$\lambda^6 - 6\lambda^4 + 10\lambda^2 - 4 = \lambda^8 - 7\lambda^6 + 14\lambda^4 - 7\lambda^2$, i.e., $\lambda^8 - 8\lambda^6 + 20\lambda^4 - 17\lambda^2 + 4 = 0$. Obviously, $2^8 + 8 \cdot 2^6 + 20 \cdot 2^4 - 17 \cdot 2^2 + 4 = 4(2^6 - 2^7 + 20 \cdot 4 - 16) = 4(64 - 128 + 80 - 16) = 0$, i.e., 2 is a root. Then $\vec{z} = (6, 4, 3, 5, 2, 4, 3, 2, 1)^T$.

Case $\tilde{A}_n$, $(n \geq 2)$.

$\tilde{A}_2$:



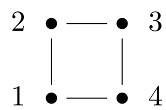
The adjacency matrix

$$[\tilde{A}_2] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and

$$[\tilde{A}_2] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$ Therefore, $r_{\tilde{A}_2} = 2$.

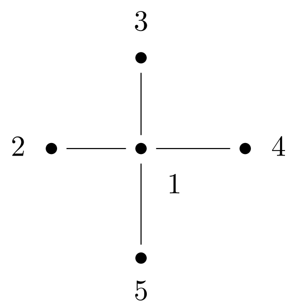
For $\tilde{A}_3$:



the adjacency matrix is $[\tilde{A}_3] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ and $[\tilde{A}_3] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Therefore, $r_{\tilde{A}_3} = 2$.

In general case, obviously, $[\tilde{A}_n]\vec{z} = 2\vec{z}$, $\vec{z} = (1, \dots, 1)^T$ and $r_{\tilde{A}_n} = 2$.

Case $\tilde{D}_4$:



Clearly, the adjacency matrix of $\tilde{D}_4$ is:

$[\tilde{D}_4] = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ and

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix}.$$

Therefore,

$z_2 + z_3 + z_4 + z_5 = \lambda z_1$;

$z_1 = \lambda z_2$;

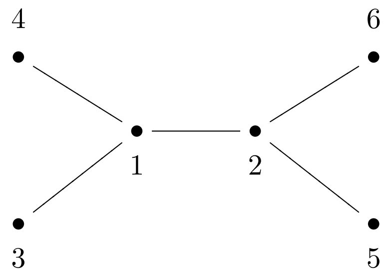
$z_1 = \lambda z_3$;

$z_1 = \lambda z_4$;

$z_1 = \lambda z_5$.

If $\lambda \leq 0$, then $\vec{z}$ is a non-positive eigenvector. So, $\lambda > 0$ and $z_2 = z_3 = z_4 = z_5$. Let $z_5 = 1$. We obtain $z_1 = \lambda$ and $\lambda^2 = 4$. Thus, $\lambda = 2$ and $\vec{z} = (2, 1, 1, 1, 1)$. We have $r_{\tilde{D}_4} = 2$.

For $\tilde{D}_5$:

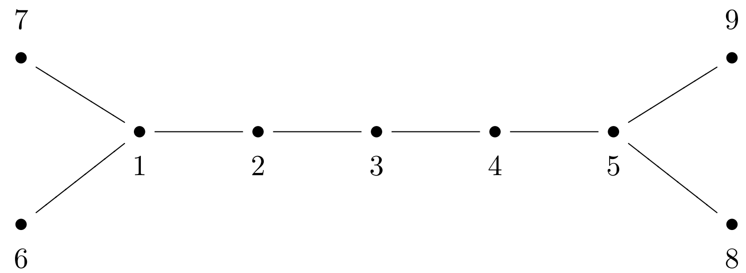


and $[\tilde{D}_5]\vec{z} = \lambda\vec{z}$, where $\vec{z} = (z_1, z_2, z_3, z_4, z_5, z_6)^T$. We have $z_5 = z_6$ and $z_3 = z_4$. $z_1 = \lambda z_3$, $z_2 = (\lambda^2 - 2)z_3$, $z_5 = \frac{\lambda^3 - 3\lambda}{2} z_3$.

Let $z_3 = 1$. Then $\vec{z} = (\lambda, \lambda^2 - 2, 1, 1, \frac{\lambda^3-3\lambda}{2}, \frac{\lambda^3-3\lambda}{2})^T$. From $z_2 = \lambda z_5$ we obtain: $\lambda^2 - 2 = \frac{\lambda^4 - 3\lambda^2}{2}$ and $\lambda^4 - 5\lambda^2 + 4 = 0$. $2^4 - 5 \cdot 4 + 4 = 0$. So, 2 is a root and

$\vec{z} = (2, 2, 1, 1, 1, 1)^T$. Therefore, $r_{\tilde{D}_5} = 2$.

Consider $\tilde{D}_8$:

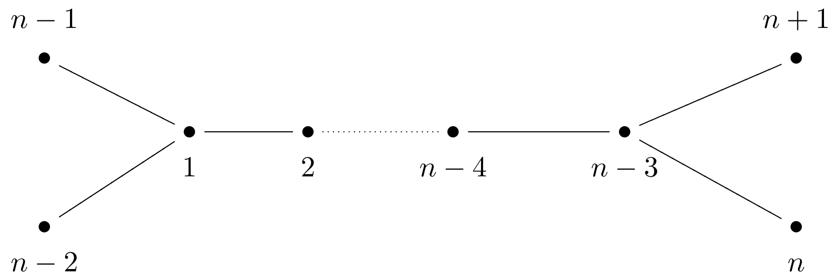


We have

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, $r_{\tilde{D}_8} = 2$.

Consider the general case $\tilde{D}_n$:



$$
\begin{array}{c|cccccccccc|}
1 & 0 & 1 & 0 & 0 & \dots & 0 & 1 & 1 & 0 & 0 \\
2 & 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\text{n-3} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 1 \\
\text{n-2} & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\text{n-1} & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\text{n} & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 \\
\text{n+1} & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 \\
\hline
 & 1 & 2 & 3 & 4 & \dots & \text{n-3} & \text{n-2} & \text{n-1} & \text{n} & \text{n+1}
\end{array}
\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ \vdots \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
= 2
\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ \vdots \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

Thus, $r_{\tilde{D}_n} = 2$.

Corollary 4.1.

(a) *For each extended Dynkin diagram* $G \in \{\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8\}$ $r_G = 2$.

(b) *For each Dynkin diagram* $G \in \{A_n, D_n, E_6, E_7, E_8\}$ *we have* $r_G < 2$.

Proof. (a) For any extended Dynkin diagram G we already gave a positive eigenvector with eigenvalue 2. Therefore, $r_G = 2$.

(b) We have the following inclusions: $A_n \subset \tilde{A}_n$, $D_n \subset \tilde{D}_n$, $E_6 \subset \tilde{E}_6$, $E_7 \subset \tilde{E}_7$, $E_8 \subset \tilde{E}_8$. By Theorem 2.1 $r_G < 2$ for any $G \in \{A_n, D_n, E_6, E_7, E_8\}$. □

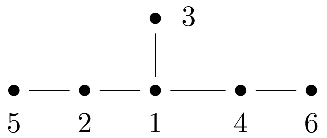
Proof of Smith's theorem. Corollary 4.1 gives the "if" part of Smith's theorem.

Conversely, let G be a connected finite simple graph with $r_G \leq 2$. If G is not a tree, then G must be the extended Dynkin diagram $\tilde{A}_n$. So, G is a tree. It is easy to see G must be a tree of the form $T_{p,q,r}$ (see [31, Exercise 4.3]). Using Theorem 2.1 we obtain that $T_{p,q,r}$ is either one of simply laced Dynkin diagrams or one of simply laced extended Dynkin diagrams.

5. Some examples

Let E_6 be given in the form:

E_6 :



Then

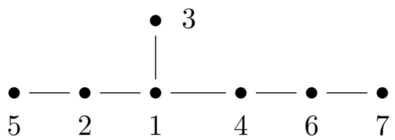
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix}$$

$$\begin{aligned} z_2 + z_3 + z_4 &= \lambda z_1 \\ z_1 + z_5 &= \lambda z_2 \\ z_1 &= \lambda z_3 \\ z_1 + z_6 &= \lambda z_4 \\ z_2 &= \lambda z_5 \\ z_4 &= \lambda z_6 \end{aligned}$$

Let $z_6 = 1$. Then $z_4 = \lambda$, $z_{)}1 = \lambda^2 - 1$ and $z_3 = \frac{\lambda^2-1}{\lambda}$. Obviously, $z_2 = \frac{\lambda_4-3\lambda^2+1}{\lambda}$. Therefore, $z_5 = \frac{\lambda^4-3\lambda^2+1}{\lambda^2}$. On the other hand, $z_5 = \lambda z_2 - z_1 = \lambda^4 - 4\lambda^2 + 2$. Consequently, $\frac{\lambda^4-3\lambda^2+1}{\lambda^2} = \lambda^4 - 4\lambda^2 + 2$. We obtain that $\lambda^6 - 5\lambda^4 + 5\lambda^2 - 1 = 0$.

Let E_7 be given as follows:

E_7 :



Then

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix}$$

$$
\begin{array}{rcl}
z_2 + z_3 + z_4 & = & \lambda z_1 \\
z_1 + z_5 & = & \lambda z_2 \\
z_1 & = & \lambda z_3 \\
z_1 + z_6 & = & \lambda z_4 \\
z_2 & = & \lambda z_5 \\
z_4 + z_7 & = & \lambda z_6 \\
z_6 & = & \lambda z_7
\end{array}
$$

Let $z_7 = 1$ and $z_6 = \lambda$. Then $z_4 = \lambda z_6 - z_7$. We obtain $z_4 = \lambda^2 - 1$. Therefore, $z_1 = \lambda^3 - 2\lambda$. Obviously, $z_3 = \lambda^2 - 2$. We have $z_2 = \lambda z_1 - z_3 - z_4$ and $z_2 = \lambda^4 - 4\lambda^2 + 3$. From the equality $z_2 = \lambda z_5$ it follows that $z_5 = \frac{\lambda^4 - 4\lambda^2 + 3}{\lambda}$. On the other hand, $z_5 = \lambda z_2 - z_1 = \lambda^5 - 5\lambda^3 + 5\lambda$. So, $\frac{\lambda^4 - 4\lambda^2 + 3}{\lambda} = \lambda^5 - 5\lambda^3 + 5\lambda$ and $\lambda^6 - 6\lambda^4 + 9\lambda^2 - 3 = 0$.

Let E_8 be given in the following form:

$$
E_8 : \quad
\begin{tikzcd}[column sep=small, row sep=small]
 & & \overset{}{\bullet}\ 3 \arrow[d, dash] & & & & \\
\underset{5}{\bullet} \arrow[r, dash] & \underset{2}{\bullet} \arrow[r, dash] & \underset{1}{\bullet} \arrow[r, dash] & \underset{4}{\bullet} \arrow[r, dash] & \underset{6}{\bullet} \arrow[r, dash] & \underset{7}{\bullet} \arrow[r, dash] & \underset{8}{\bullet}
\end{tikzcd}
$$

Then

$$
\begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \end{pmatrix}
= \lambda
\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \end{pmatrix}
$$

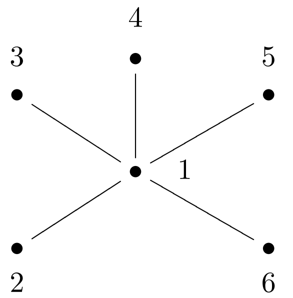
We obtain the following system of linear equations

$$
\begin{array}{rcl}
z_2 + z_3 + z_4 & = & \lambda z_1 \\
z_1 + z_5 & = & \lambda z_2 \\
z_1 & = & \lambda z_3 \\
z_1 + z_6 & = & \lambda z_4 \\
z_2 & = & \lambda z_5 \\
z_4 + z_7 & = & \lambda z_6 \\
z_6 + z_8 & = & \lambda z_7 \\
z_7 & = & \lambda z_8
\end{array}
$$

Let $z_8 = 1$. Then $z_7 = \lambda$ and $z_6 = \lambda^2 - 1$. Obviously, we have: $z_4 = \lambda^3 - 2\lambda$, $z_1 = \lambda^4 - 3\lambda^2 + 1$, $z_3 = \frac{\lambda^4 - 3\lambda^2 + 1}{\lambda}$, $z_2 = \frac{\lambda^6 - 5\lambda^4 + 6\lambda^2 - 1}{\lambda}$ and $z_5 = \frac{\lambda^6 - 5\lambda^4 + 6\lambda^2 - 1}{\lambda}$.

Hence, $z_5 = \lambda z_2 - z_1 = \lambda^6 - 6\lambda^4 + 9\lambda^2 - 2$. Consequently, $\lambda^6 - 6\lambda^4 + 9\lambda^2 - 2 = \frac{\lambda^6 - 5\lambda^4 + 6\lambda^2 - 1}{\lambda^2}$ and $\lambda^8 - 7\lambda^6 + 14\lambda^4 - 8\lambda^2 + 1 = 0$

In conclusion we consider the following simple graph G_5:



with the adjacency matrix $[G_5]$:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We have

$$[G_5]\vec{z} = \lambda \vec{z}, \tag{1}$$

where $\vec{z} = (z_1, z_2, z_3, z_4, z_5, z_6)^T$. From 1 we obtain

$$\begin{array}{rcl} z_2 + z_3 + z_4 + z_5 + z_6 & = & \lambda z_1 \\ z_1 & = & \lambda z_2 \\ z_1 & = & \lambda z_3 \\ z_1 & = & \lambda z_4 \\ z_1 & = & \lambda z_5 \\ z_1 & = & \lambda z_6 \end{array}.$$

Consequently, $5z_1 = \lambda(z_2 + z_3 + z_4 + z_5 + z_6) = \lambda^2 z_1$. Since, $z_1 \neq 0$, we obtain $\lambda = \sqrt{5}$ and $\vec{z} = (\sqrt{5}, 1, 1, 1, 1, 1)$ and $r_{G_5} = \sqrt{5} > 2$.

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