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Exceptional collections of sheaves on quadrics in positive characteristic*

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Abstract. On quadrics in large positive characteristic we construct an exceptional collection of sheaves from the G_1P-Verma module associated to the Frobenius direct image of the structure sheaf of the quadric. They are all locally free of finite rank and defined over $\mathbb{Z}$, producing Kapranov's collection over $\mathbb{C}$ by base change.

We also determine on a general homogeneous projective variety in large positive characteristic a direct summand of the Frobenius direct image of the structure sheaf, which is "dual" to the celebrated Frobenius splitting of Mehta and Ramanathan.

Introduction

Since the seminal work of Beilinson [Be] much has been done on exceptional collections of sheaves on complex smooth projective varieties. They bridge between the geometric categories of the coherent sheaves on the varieties and the algebraic categories of the modules of finite type over the endomorphism algebras of the sum of the sheaves of the collections.

Each complex smooth projective variety admits a $\mathbb{Z}$-form $G_{\mathbb{Z}}/P_{\mathbb{Z}}$ with $G_{\mathbb{Z}}$ a Chevalley $\mathbb{Z}$-group scheme and $P_{\mathbb{Z}}$ a parabolic subgroup scheme. In particular, for a quadric $\mathcal{Q}_{\mathbb{C}}$ one can take $G_{\mathbb{Z}}$ to be the simply connected cover of a special orthogonal group. In this paper we construct Kapranov's exceptional collection on $\mathcal{Q}_{\mathbb{C}}$ using representation theory of G_1P-Verma module $\hat{\nabla}_P(\Bbbk)$, where G (resp. P) is the algebraic group over an algebraically closed field $\Bbbk$ of positive characteristic $p \neq 2$ obtained from $G_{\mathbb{Z}}$ (resp. $P_{\mathbb{Z}}$) by base change and G_1 is the Frobenius kernel of G. The module $\hat{\nabla}_P(\Bbbk)$ is of finite dimension $p^{\dim \mathcal{Q}_{\mathbb{C}}}$. Let W (resp. W_P) be the Weyl group of G (resp. P) and W^P be the set of coset representatives of W/W_P

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having minimal length. Let ρ denote a half sum of the positive roots of G, $w\bullet 0 = w\rho - \rho$, and $(w\bullet 0)^1$ be the p-part of $w\bullet 0$, i.e., $w\bullet 0 = (w\bullet 0)^0 + p(w\bullet 0)^1$ with $\langle (w\bullet 0)^0, \alpha^\vee\rangle \in [0, p[$ for each simple coroot $\alpha^\vee$ of G. Assume for the moment that $p \geq h$ the Coxeter number of G. We show

Theorem: . *For each $w \in W^P$ let $\nabla^P(w^{-1}\bullet(w\bullet 0)^1)$ be the P-module induced from 1-dimensional B-module afforded by $w^{-1}\bullet(w\bullet 0)^1$ and let $\mathcal{E}_w = \mathcal{L}(\nabla^P(w^{-1}\bullet(w\bullet 0)^1))$ be the locally free sheaf on the quadric $\mathcal{Q} = G/P$ associated to $\nabla^P(w^{-1}\bullet(w\bullet 0)^1)$.*

(i) All $\nabla^P(w^{-1}\bullet(w\bullet 0)^1)$, $w \in W^P$, are 1-dimensional except for 1 (resp. 2) if $\dim \mathcal{Q}$ is odd (resp. even), in which case $\nabla^P(w^{-1}\bullet(w\bullet 0)^1)$ are the spinor modules.

(ii) The $\mathcal{E}_w$, $w \in W^P$, form an exceptional collection on $\mathcal{Q}$ such that

(1) *$\forall w \in W^P$, $\mathbf{Mod}_{\mathcal{Q}}(\mathcal{E}_w, \mathcal{E}_w) = \Bbbk$,*
(2) *$\forall x, y \in W^P$, $\mathrm{Ext}^i_{\mathcal{Q}}(\mathcal{E}_x, \mathcal{E}_y) = 0$ if $i > 0$ while $\mathbf{Mod}_{\mathcal{Q}}(\mathcal{E}_x, \mathcal{E}_y) \neq 0$ iff $x \geq y$ in the Chevalley-Bruhat order.*
(3) *The smallest triangulated subcategory of the bounded derived category $\mathrm{D}^b(\mathrm{coh}(\mathcal{Q}))$ of coherent sheaves on $\mathcal{Q}$ containing all $\mathcal{E}_w$, $w \in W^P$, is the whole of $\mathrm{D}^b(\mathrm{coh}(\mathcal{Q}))$.*

Thus, by Belinson's lemma [HB, Th. 7.6], $\mathrm{R}\mathbf{Mod}_{\mathcal{Q}}(\coprod_{w\in W^P}\mathcal{E}_w, ?)$ gives a triangulated equivalence from $\mathrm{D}^b(\mathrm{coh}(\mathcal{Q}))$ to the bounded derived category of the modules of finite type over the opposite algebra of the endomorphism algebra of $\coprod_{w\in W^P}\mathcal{E}_w$. All our $\mathcal{E}_w$'s are defined over $\mathbb{Z}$, and yield Kapranov's collection over $\mathcal{Q}_{\mathbb{C}}$ by base change. We actually obtain a stronger statement, see (3.7).

The organization of the paper is as follows: in §1 we describe basic structures of $\hat{\nabla}_P(\Bbbk)$ regarded as a G_1T-module, T a maximal torus of P, for general semisimple group G, and explain how our modules $\nabla^P(w^{-1}\bullet(w\bullet 0)^1)$ arise in $\hat{\nabla}_P(\Bbbk)$. As Haastert [Haa] observed, the sheaf $\mathcal{L}_{G/G_1P}(\hat{\nabla}_P(\Bbbk))$ on G/G_1P associated to the G_1P-module $\hat{\nabla}_P(\Bbbk)$ is isomorphic to the direct image $F_*\mathcal{O}_{G/P}$ of the structure sheaf $\mathcal{O}_{G/P}$ of G/P under the Frobenius endomorphism F of G/P through the isomorphism of schemes $G/G_1P \to G/P$. Langer [La] has completely determined the decomposition of $F_*\mathcal{O}_{\mathcal{Q}}$ into indecomposables in any positive characteristic $p > 2$, and showed that $F_*\mathcal{O}_{G/P}$ is tilting for $p \geq h$. Our $\mathcal{E}_w$, $w \in W^P$, exhaust the pairwise non-isomorphic indecomposable direct summands of $F_*\mathcal{O}_{\mathcal{Q}}$ for $p \geq h$. Also, on general G/P the fact that $\mathcal{O}_{G/P}$ is a direct summand of $F_*\mathcal{O}_{G/P}$, discovered by Mehta and Ramanathan [MR], is very important and has brought a breakthrough in the study of the geometry of G/P, cf. [BK]. In our point of view the splitting is induced by the socle of $\hat{\nabla}_P(\Bbbk)$. For $p \geq 4h-5$ we will determine another direct summand of $F_*\mathcal{O}_{G/P}$ in (1.7) arising from the head of $\hat{\nabla}_P(\Bbbk)$, which may be of independent interest.

After stating the definition of quadrics in §2 and some basics of special orthogonal groups in §3, we compute in §4 the extensions between the $\mathcal{E}_w$'s, using representation theory, especially the fact that the highest weight of the spinor representation is minuscule. We also make use of crystal bases from the theory of quantum groups.

In the final section of the paper we will include a brief account of the generation of $\mathrm{D}^b(\mathrm{coh}(\mathcal{Q}))$ by the $\mathcal{E}_w$'s, which was missing in [La], to convince the reader that Kapranov's resolution of the diagonal direct image of the structure sheaf on a quadric holds independent of characteristic $\neq 2$.

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1° Structure of G_1P-Verma modules

We fix an algebraically closed field $\Bbbk$ of positive characteristic p. In this section G will denote a general semisimple algebraic group over $\Bbbk$ and P a parabolic subgroup of G, B a Borel subgroup of P, and T a maximal torus of B. We assume that G is simply connected.

(1.1) Let G_1 be the scheme theoretic kernel of the Frobenius endomorphism of G. Let Λ denote the character group of T, $R \subset \Lambda$ the set of roots of G relative to T. We choose a positive system R^+ of R so that that the roots of B are negative. Let $\Lambda^+ = \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in R^+\}$ be the set of dominant weights. Denoting by R^s the set of simple roots, let ϖ_α be the fundamental weight associated to $\alpha \in R^s$. For each $\lambda \in \Lambda$ we write $\lambda = \lambda^0 + p\lambda^1$ with $\lambda^0 \in \Lambda^+$ such that $\langle \lambda^0, \alpha^\vee \rangle < p \ \forall \alpha \in R^s$. The simple G-modules are parametrized by Λ^+; we denote the simple G-module of highest weight $\lambda \in \Lambda^+$ by $L(\lambda)$. Each $L(\lambda^0)$ remains simple for G_1, and all simple G_1-modules are obtained thus. We equip Λ with the standard partial order: $\lambda \leq \mu$ iff $\mu - \lambda \in \sum_{\alpha \in R^+} \mathbb{N}\alpha$. Let W (resp. W_P) be the Weyl group of G (resp. P) and let ℓ be the length function on W with respect to the the simple reflections s_α, $\alpha \in R^s$. Let w_0 (resp. w_P) be the longest element of W (resp. W_P). Let also $W^P = \{x \in W \mid \ell(xy) = \ell(x) + \ell(y) \ \forall y \in W_P\}$, so $W = \sqcup_{x \in W^P} xW_P$. Put $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$. For $w \in W$ and $\lambda \in \Lambda$ we write $w \bullet \lambda = w(\lambda + \rho) - \rho$.

Let $\hat{\nabla}_P = \mathrm{ind}_P^{G_1P} = \mathbf{Sch}_{\Bbbk}(G_1P, ?)^P$ be the induction functor from the category of P-modules to the category of G_1P-modules. We will often write $\hat{\nabla}_P(\Bbbk)$ for $\hat{\nabla}_P(0)$ induced from the trivial 1-dimensional P-module of weight 0. For each $\lambda \in \Lambda$ let $\hat{L}(\lambda)$ be the simple G_1T-module of highest weight λ: $\hat{L}(\lambda) = L(\lambda^0) \otimes p\lambda^1$. Let us denote by $[M : L]$ the multiplicity of G_1T-simple L in a composition series of G_1T-module M.

(1.2) Let L_P denote the standard Levi subgroup of P and let $R_L \subseteq R$ be the set of roots for L_P with the positive system $R_L^+ = R_L \cap R^+$. Put $\rho_L = \frac{1}{2}\sum_{\alpha \in R_L^+} \alpha$. In [AbK] we showed that the G_1T-head of $\hat{\nabla}_P(\Bbbk)$ is $\hat{L}(w^P \bullet 0) \otimes p\{-2\rho_P + w_0(-2\rho_P)^1 - (-2\rho_P)^1\}$ with $w^P = w_0 w_P$ and $\rho_P = \rho - \rho_L$. Write $2\rho_P = \sum_{\alpha \in R^s} n_\alpha \varpi_\alpha$, $n_\alpha \in \mathbb{Z}$. One has $n_\alpha = 0\ \forall \alpha \in R_L$ and $n_\alpha \in [2, h]$ if $\alpha \notin R_L$ [AbK, 1.1].

Lemma: . *Let $r_\alpha \in \mathbb{N}$ such that $r_\alpha p - n_\alpha \in [0, p[\ \forall \alpha \in R^s$. Then*

$$\begin{aligned}
\mathrm{hd}_{G_1T}\hat{\nabla}_P(\Bbbk) &= \hat{L}(w^P \bullet 0) \otimes p\{-2\rho_P + w_0(-2\rho_P)^1 - (-2\rho_P)^1\} \\
&= L((w^P \bullet 0)^0) \otimes p(\sum_{\alpha \in R^s \setminus R_L} r_\alpha \varpi_\alpha - 2\rho_P) \\
&= L((w^P \bullet 0)^0) \otimes p\{(w^P)^{-1} \bullet (w^P \bullet 0)^1\}
\end{aligned}$$

Proof: One has

$$\begin{aligned}
w^P \bullet 0 &= w^P\rho - \rho = w^P(\rho_L + \rho_P) - \rho = w_0 w_P(\rho_L + \rho_P) - \rho \\
&= w_0(-\rho_L + \rho_P) - \rho = -w_0\rho_L + w_0\rho_P + w_0(\rho_L + \rho_P) = 2w_0\rho_P \\
&= w_0 \sum_{\alpha \in R^s \setminus R_L} n_\alpha \varpi_\alpha = \sum_{\alpha \in R^s \setminus R_L} (-n_\alpha)\varpi_{-w_0\alpha}.
\end{aligned} \tag{1}$$

Thus

$$(w^P \bullet 0)^0 = \sum_{\alpha \in R^s \setminus R_L} (r_\alpha p - n_\alpha)\varpi_{-w_0\alpha}$$

and $(w^P \bullet 0)^1 = -\sum_{\alpha \in R^s \setminus R_L} r_\alpha \varpi_{-w_0\alpha}$. We are to show

$$\begin{aligned}
(w^P \bullet 0)^1 - 2\rho_P + w_0(-2\rho_P)^1 - (-2\rho_P)^1 &= \sum_{\alpha \in R^s \setminus R_L} r_\alpha \varpi_\alpha - 2\rho_P \\
&= (w^P)^{-1} \bullet (w^P \bullet 0)^1.
\end{aligned}$$

As $-2\rho_P = -\sum_{\alpha\in R^s\setminus R_L} n_\alpha\varpi_\alpha$, $(-2\rho_P)^0 = \sum_{\alpha\in R^s\setminus R_L}(r_\alpha p - n_\alpha)\varpi_\alpha = p\sum_{\alpha\in R^s\setminus R_L} r_\alpha\varpi_\alpha - 2\rho_P$ and $(-2\rho_P)^1 = -\sum_{\alpha\in R^s\setminus R_L} r_\alpha\varpi_\alpha$. Then

$$\begin{aligned}
&(w^P\bullet 0)^1 - 2\rho_P + w_0(-2\rho_P)^1 - (-2\rho_P)^1\\
&\qquad = -\sum_{\alpha\in R^s\setminus R_L} r_\alpha\varpi_{-w_0\alpha} - 2\rho_P + w_0\sum_{\alpha\in R^s\setminus R_L}(-r_\alpha\varpi_\alpha) + \sum_{\alpha\in R^s\setminus R_L} r_\alpha\varpi_\alpha\\
&\qquad = \sum_{\alpha\in R^s\setminus R_L} r_\alpha\varpi_\alpha - 2\rho_P
\end{aligned}$$

while

$$\begin{aligned}
(w^P)^{-1}\bullet(w^P\bullet 0)^1 &= w_P w_0\bullet\Big(-\sum_{\alpha\in R^s\setminus R_L} r_\alpha\varpi_{-w_0\alpha}\Big)\\
&= w_P w_0\Big(\rho - \sum_{\alpha\in R^s\setminus R_L} r_\alpha\varpi_{-w_0\alpha}\Big) - \rho\\
&= w_P\Big(-\rho + \sum_{\alpha\in R^s\setminus R_L} r_\alpha\varpi_\alpha\Big) - \rho\\
&= -w_P\rho - \rho + \sum_{\alpha\in R^s\setminus R_L} r_\alpha\varpi_\alpha\\
&= -2\rho_P + \sum_{\alpha\in R^s\setminus R_L} r_\alpha\varpi_\alpha.
\end{aligned}$$

(1.3) Recall from [J, II.9.16.4] that for any $\lambda\in W$ and $w\in W$

$$\begin{aligned}
[\hat\nabla_B(w\bullet\lambda) : L(\mu^0)\otimes pw\bullet\mu^1] &= [\hat\nabla_B(w\bullet\lambda) : \hat L(\mu^0 + pw\bullet\mu^1)] \qquad (1)\\
&= [\hat\nabla_B(\lambda) : \hat L(\mu)] = [\hat\nabla_B(\lambda) : L(\mu^0)\otimes p\mu^1].
\end{aligned}$$

Proposition: . *For each $w\in W^P$ one has $[\hat\nabla_P(\Bbbk) : L((w\bullet 0)^0))\otimes p(w^{-1}\bullet (w\bullet 0)^1)] = 1$.*

Proof: For each $w\in W$ one has

$$\begin{aligned}
&[\hat\nabla_B(\Bbbk) : L((w\bullet 0)^0))\otimes p(w^{-1}\bullet(w\bullet 0)^1)]\\
&\qquad = [\hat\nabla_B(w^{-1}\bullet(w\bullet 0)) : \hat L((w\bullet 0)^0 + p(w^{-1}\bullet(w\bullet 0)^1))]\\
&\qquad = [\hat\nabla_B(w\bullet 0) : \hat L((w\bullet 0)^0 + p(w\bullet 0)^1)] \quad \text{by (1)}\\
&\qquad = [\hat\nabla_B(w\bullet 0) : \hat L(w\bullet 0)] = 1.
\end{aligned}$$

On the other hand, if $w\in W^P$, $L((w\bullet 0)^0))\otimes p(w^{-1}\bullet(w\bullet 0)^1)$ does appear as a composition factor of $\hat\nabla_P(\Bbbk)$, cf. [AbK, 5.3].

(1.4) **Lemma:** *$\forall \gamma \in \sum_{\alpha \in R_L^+} \mathbb{N}\alpha \setminus 0$, $L((w \bullet 0)^0) \otimes p(w^{-1} \bullet (w \bullet 0)^1 + \gamma)$ is not a G_1T-composition factor of $\hat{\nabla}_P(\Bbbk)$.*

Proof: Just suppose it is. One would then have

$$
\begin{aligned}
0 &\neq [\hat{\nabla}_B(\Bbbk) : L((w \bullet 0)^0) \otimes p(w^{-1} \bullet (w \bullet 0)^1 + \gamma)] \\
&= [\hat{\nabla}_B(w^{-1} \bullet (w \bullet 0)) : \hat{L}((w \bullet 0)^0 + p(w^{-1} \bullet (w \bullet 0)^1 + \gamma))] \\
&= [\hat{\nabla}_B(w^{-1} \bullet (w \bullet 0)) : \hat{L}((w \bullet 0)^0 + p(w^{-1} \bullet ((w \bullet 0)^1 + w\gamma)))] \\
&= [\hat{\nabla}_B(w \bullet 0) : \hat{L}(w \bullet 0 + w\gamma))]
\end{aligned}
$$

with $w\gamma > 0$, contradicting the fact that $w \bullet 0$ is the highest weight of $\hat{\nabla}_B(w \bullet 0)$.

(1.5) Let $\nabla(?) = \mathrm{ind}_B^G = \mathbf{Sch}_\Bbbk(G, ?)^B$ denote the induction functor from the category of B-modules to that of G-modules, and define $\nabla_P(?) = \mathrm{ind}_P^G = \mathbf{Sch}_\Bbbk(G, ?)^P$, $\nabla^P(?) = \mathrm{ind}_B^P = \mathbf{Sch}_\Bbbk(P, ?)^B$ likewise. Let $\Lambda_L^+ = \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in R_L^+\}$. Let also $\Lambda_P = \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle = 0 \ \forall \alpha \in R_L\}$. Thus $\nabla^P(\lambda) = \lambda$ for $\lambda \in \Lambda_P$.

Lemma: . *(i) $\forall w \in W \ \forall \alpha \in R^s$, $\langle w \bullet 0, \alpha^\vee \rangle \in [-h, h-2]$.*

(ii) Assume $p \geq h$ and let $w \in W^P$. Then $w^{-1} \bullet (w \bullet 0)^1 \in \Lambda_L^+$ with $\langle w^{-1} \bullet (w \bullet 0)^1 + \rho_L, \beta^\vee \rangle < p \ \forall \beta \in R_L^+$. The P-module $\nabla^P(w^{-1} \bullet (w \bullet 0)^1)$ is simple.

Proof: (i) One has $\langle w \bullet 0, \alpha^\vee \rangle = \langle w \bullet 0 + \rho, \alpha^\vee \rangle - 1 = \langle w\rho, \alpha^\vee \rangle - 1 = \langle \rho, w^{-1}\alpha^\vee \rangle - 1$. Let $\alpha_0^\vee$ denote the highest coroot of R. If $w^{-1}\alpha > 0$, $\langle w \bullet 0, \alpha^\vee \rangle \leq \langle \rho, \alpha_0^\vee \rangle - 1 = h - 2$ whereas if $w^{-1}\alpha < 0$, $\langle w \bullet 0, \alpha^\vee \rangle \geq -\langle \rho, \alpha_0^\vee \rangle - 1 = -h$.

(ii) Write $w \bullet 0 = \sum_{\alpha \in R^s} c_\alpha \varpi_\alpha$, $c_\alpha \in \mathbb{Z}$. As $c_\alpha \in [-p, p[\ \forall \alpha$ by (i) under the hypothesis, $(w \bullet 0)^0 = \sum_{c_\alpha \geq 0} c_\alpha \varpi_\alpha + \sum_{c_\alpha < 0} (p + c_\alpha)\varpi_\alpha$ and $(w \bullet 0)^1 = -\sum_{c_\alpha < 0} \varpi_\alpha$. $\forall \beta \in R_L^+$,

$$
\begin{aligned}
\langle w^{-1} \bullet (w \bullet 0)^1 + \rho_L, \beta^\vee \rangle &= \langle w^{-1} \sum_{c_\alpha \geq 0} \varpi_\alpha - \rho + \rho_L, \beta^\vee \rangle \\
&= \langle w^{-1} \sum_{c_\alpha \geq 0} \varpi_\alpha, \beta^\vee \rangle \quad \text{by [AbK, 1.1]} \\
&= \langle \sum_{c_\alpha \geq 0} \varpi_\alpha, w\beta^\vee \rangle \in [0, h[.
\end{aligned}
$$

The last assertion follows from Andersen's strong linkage principle [J, II.6.16].

(1.6) Assume in this subsection that $p > h$. Then the principal block for G_1 is in bijective correspondence to W, so the simple G_1-modules in this block may be written $L(w)$, abbreviating $L((w \bullet 0)^0)$. If $\mathrm{soc}_i \hat{\nabla}_P(\Bbbk)$ denotes the i-th G_1T-semisimple subquotient of the socle series of $\hat{\nabla}_P(\Bbbk)$, one can write

$$\mathrm{soc}_i \hat{\nabla}_P(\Bbbk) = \coprod_{w \in W} L(w) \otimes G_1\mathbf{Mod}(L(w), \mathrm{soc}_i \hat{\nabla}_P(\Bbbk)).$$

As $\hat{\nabla}_P(\Bbbk)$ is a G_1P-module, the direct sum decomposition above holds as G_1P-modules [J, I.6.16]. As G_1 acts trivially on the multiplicity space $G_1\mathbf{Mod}(L(w), \mathrm{soc}_i \hat{\nabla}_P(\Bbbk))$ of $L(w)$ in $\mathrm{soc}_i \hat{\nabla}_P(\Bbbk)$, there is a P-module M such that $M^{[1]} = G_1\mathbf{Mod}(L(w), \mathrm{soc}_i \hat{\nabla}_P(\Bbbk))$; $M^{[1]}$ is M as a $\Bbbk$-linear space with P acting through the Frobenius on P. We will denote M by $\mathrm{soc}^1_{i,w}$.

For p large enough Lusztig's conjecture on the irreducible characters for G_1T is now a theorem [J, C]/[F]. Assuming the Lusztig conjecture, we showed in [AbK] that $\hat{\nabla}_P(\Bbbk)$ as a G_1T-module can be equipped with a $\mathbb{Z}$-gradation, and hence is rigid as follows from the general principle [BGS], of Loewy length $\ell(w^P) + 1$, and that each $L(w) \otimes p(w^{-1} \bullet (w \bullet 0)^1)$, $w \in W^P$, appears in the $(\ell(w) + 1)$-st subquotient of the G_1T-socle series of $\hat{\nabla}_P(\Bbbk)$. As each root subgroup U_α, $\alpha \in R_L^+$, of L_P fixes a vector of weight $w^{-1} \bullet (w \bullet 0)^1$ in the P-module $\mathrm{soc}^1_{\ell(w)+1,w}$, $w \in W^P$, by (1.4), the P-submodule generated by that vector has simple head of highest weight $w^{-1} \bullet (w \bullet 0)^1$, which coincides with $\nabla^P(w^{-1} \bullet (w \bullet 0)^1)$ by (1.5), and hence $\nabla^P(w^{-1} \bullet (w \bullet 0)^1)$ is a P-subquotient of $\mathrm{soc}^1_{\ell(w)+1,w}$. In particular, if $w^{-1} \bullet (w \bullet 0)^1 \in \Lambda_P$, $\nabla^P(w^{-1} \bullet (w \bullet 0)^1) = w^{-1} \bullet (w \bullet 0)^1$ is a 1-dimensional submodule of $\mathrm{soc}^1_{\ell(w)+1,w}$.

Consider a commutative diagram of schemes

$$\begin{tikzcd} G/P \arrow[r, "F"] \arrow[d] & G/P \\ G/G_1P \arrow[ur, "\sim"'] & \end{tikzcd}$$

with $G/P \to G/G_1P$ the natural morphism. Thus

$$F_*\mathcal{O}_{G/P} \simeq \mathcal{L}_{G/G_1P}(\hat{\nabla}_P(\Bbbk))$$

admits a filtration with subquotients

$$\mathcal{L}(\mathrm{soc}_i\hat{\nabla}_P(\Bbbk)) = \coprod_{w\in W} L(w)\otimes\mathcal{L}_{G/G_1P}((\mathrm{soc}^1_{i,w})^{[1]}) = \coprod_{w\in W} L(w)\otimes\mathcal{L}_{G/P}(\mathrm{soc}^1_{i,w})$$

and hence $\mathcal{L}_{G/P}(\nabla^P(w^{-1}\bullet(w\bullet 0)^1))$ is a subquotient of $F_*\mathcal{O}_{G/P}$.

(1.7) We showed in [K94] that the Frobenius comorphism $F^\sharp : \mathcal{O}_{G/P} \to F_*\mathcal{O}_{G/P}$ induced by the inclusion $\Bbbk = \mathrm{soc}\hat{\nabla}_P(\Bbbk) \to \hat{\nabla}_P(\Bbbk)$ admits a left inverse, a famous fact found originally by Mehta and Ramanathan [MR]. Let now $\pi : \hat{\nabla}_P(\Bbbk) \to \mathrm{hd}_{G_1P}\hat{\nabla}_P(\Bbbk) = L((w^P\bullet 0)^0)\otimes p\{(w^P)^{-1}\bullet(w^P\bullet 0)^1\}$ be the quotient and put $\bar{\mathcal{P}} = G/G_1P$. We ask if $\mathcal{L}_{\bar{\mathcal{P}}}(\pi)$ admits a right inverse, equivalently, if

$$\begin{aligned}&\mathbf{Mod}_{\bar{\mathcal{P}}}(\mathcal{L}_{\bar{\mathcal{P}}}(L((w^P\bullet 0)^0)\otimes p(w^P)^{-1}\bullet(w^P\bullet 0)^1),\mathcal{L}_{\bar{\mathcal{P}}}(\pi)) :\\ &\quad\mathbf{Mod}_{\bar{\mathcal{P}}}(\mathcal{L}_{\bar{\mathcal{P}}}(L((w^P\bullet 0)^0)\otimes p(w^P)^{-1}\bullet(w^P\bullet 0)^1),\mathcal{L}_{\bar{\mathcal{P}}}(\hat{\nabla}_P(\Bbbk))) \to\\ &\mathbf{Mod}_{\bar{\mathcal{P}}}(\mathcal{L}_{\bar{\mathcal{P}}}(L((w^P\bullet 0)^0)\otimes p(w^P)^{-1}\bullet(w^P\bullet 0)^1),\mathcal{L}_{\bar{\mathcal{P}}}(L((w^P\bullet 0)^0)\otimes p(w^P)^{-1}\bullet(w^P\bullet 0)^1))\end{aligned}$$

is surjective. The latter reads

$$\begin{aligned}&\mathrm{ind}^G_{G_1P}(L((w^P\bullet 0)^0)^*\otimes(-p(w^P)^{-1}\bullet(w^P\bullet 0)^1)\otimes\pi) :\\ &\quad\mathrm{ind}^G_{G_1P}(L((w^P\bullet 0)^0)^*\otimes(-p(w^P)^{-1}\bullet(w^P\bullet 0)^1)\otimes\hat{\nabla}_P(\Bbbk)) \twoheadrightarrow\\ &\mathrm{ind}^G_{G_1P}(L((w^P\bullet 0)^0)^*\otimes(-p(w^P)^{-1}\bullet(w^P\bullet 0)^1)\otimes L((w^P\bullet 0)^0)\otimes p(w^P)^{-1}\bullet(w^P\bullet 0)^1),\end{aligned}$$

and hence is equivalent to

$$\begin{tikzcd}
\mathrm{ind}^G_{G_1P}(-p(w^P)^{-1}\bullet(w^P\bullet 0)^1\otimes\hat{\nabla}_P(\Bbbk)) \arrow[r, "\sim"] \arrow[dd, "\mathrm{ind}^G_{G_1P}(-p(w^P)^{-1}\bullet(w^P\bullet 0)^1\otimes\pi)"'] & \nabla_P(-p(w^P)^{-1}\bullet(w^P\bullet 0)^1) \arrow[d, "\sim"] \\
\arrow[r, phantom, "\circlearrowleft"] & \nabla(-p(w^P)^{-1}\bullet(w^P\bullet 0)^1) \arrow[d, dotted] \\
\mathrm{ind}^G_{G_1P}(L((w^P\bullet 0)^0)) \arrow[r, "\sim"'] & L((w^P\bullet 0)^0).
\end{tikzcd}$$

In the case $P = B$ one has $(w^P\bullet 0)^0 = (p-2)\rho$ and $-p(w^P)^{-1}\bullet(w^P\bullet 0)^1 = p\rho$, and hence $\mathrm{ind}^G_{G_1B}(p\rho\otimes\pi)$ is certainly surjective for $p \geq h$ [GK, 7.13]. In general, the assertion is equivalent to that $\mathrm{ind}_{G_1P}^G(-p(w^P)^{-1} \bullet (w^P \bullet 0)^1 \otimes \pi) \neq 0$. Put $\eta = -(w^P)^{-1} \bullet (w^P \bullet 0)^1$ and $M = \hat{\nabla}_B(p\eta)/\ker(p\eta \otimes \pi)$. One has a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{ind}_{G_1P}^G(\ker(p\eta \otimes \pi)) \arrow[r] \arrow[d] & \nabla_P(p\eta) \arrow[r, "\mathrm{ind}_{G_1P}^G(p\eta\otimes\pi)"] \arrow[d, "\sim"] & L(w^P) \arrow[d, hook] \\
0 \arrow[r] & \mathrm{ind}_{G_1B}^G(\ker(p\eta \otimes \pi)) \arrow[r] & \nabla(p\eta) \arrow[r] & \mathrm{ind}_{G_1B}^G(M) \arrow[d] \\
 & & & \mathrm{R}^1\mathrm{ind}_{G_1B}^G(\ker(p\eta \otimes \pi))
\end{tikzcd}$$

If $\mathrm{ind}_{G_1P}^G(p\eta \otimes \pi) = 0$, $L(w^P) \leq \mathrm{R}^1\mathrm{ind}_{G_1B}^G(\ker(p\eta \otimes \pi))$. But the only G_1B-composition factors of $\ker(p\eta \otimes \pi)$ contributing to $L(w^P)$ as a factor of $\mathrm{R}^1\mathrm{ind}_{G_1B}^G(\ker(p\eta \otimes \pi))$ are $L(w^P) \otimes p\mu$, $\mu \in \Lambda$, by Steinberg's tensor product theorem; $\mathrm{R}^1\mathrm{ind}_{G_1B}^G(L(w^P) \otimes p\mu) \simeq L(w^P) \otimes \mathrm{R}^1\mathrm{ind}_B^G(\mu)^{[1]} = \mathrm{H}^1(G/B, \mathcal{L}(\mu))^{[1]}$. One has

$$\begin{aligned}
[\hat{\nabla}_B(p\eta) : L(w^P)] &= [\hat{\nabla}_B(-p(w^P)^{-1} \bullet (w^P \bullet 0)^1) : L(w^P)] \\
&= [\hat{\nabla}_B(\Bbbk) : \hat{L}((w^P \bullet 0)^0 + p(w^P)^{-1} \bullet (w^P \bullet 0)^1)] \\
&= [\hat{\nabla}_B(w^P \bullet 0) : \hat{L}((w^P \bullet 0)^0 + p(w^P \bullet 0)^1)] \\
&= [\hat{\nabla}_B(w^P \bullet 0) : \hat{L}(w^P \bullet 0)] = 1
\end{aligned}$$

while

$$\begin{aligned}
[\hat{\nabla}_B(p\eta) : L(w^P) \otimes p\mu] &= [\hat{\nabla}_B(-p(w^P)^{-1} \bullet (w^P \bullet 0)^1) : L(w^P) \otimes p\mu] \qquad (1) \\
&= [\hat{\nabla}_B(\Bbbk) : \hat{L}((w^P \bullet 0)^0 + p(w^P)^{-1} \bullet (w^P \bullet 0)^1 + p\mu)] \\
&= [\hat{\nabla}_B(w^P \bullet 0) : \hat{L}((w^P \bullet 0)^0 + p((w^P \bullet 0)^1 + w^P\mu))] \\
&= [\hat{\nabla}_B(w^P \bullet 0) : \hat{L}(w^P \bullet 0 + pw^P\mu)].
\end{aligned}$$

Suppose that $[\hat{\nabla}_B(p\eta) : L(w^P) \otimes p\mu] \neq 0$. By (1) we must have $w^P\mu \leq 0$. Also, one has $|\langle w^P\mu + \rho, \alpha^\vee \rangle| < 2(h-1)$ $\forall \alpha \in R$ by [A, 1.7]. Thus for $p \geq 4h-5$ we will have $|\langle \mu + \rho, \alpha^\vee \rangle| \leq p$ $\forall \alpha \in R$ and Bott's theorem holds for $\mathrm{H}^\bullet(G/B, \mathcal{L}(\mu))$ [J, II.5.5]. If $\mathrm{H}^1(G/B, \mathcal{L}(\mu)) \neq 0$ to contribute to forming $L(w^P)$, we must have $\mu = s_\alpha \bullet 0 = -\alpha$ for some $\alpha \in R^s$. As $0 \geq w^P(-\alpha) = w_0 w_P(-\alpha)$, $w_P\alpha < 0$, and hence $\alpha \in R_L^+$. Then $w^P(-\alpha) = -\beta$ for some $\beta \in R^s$. Thus

$$0 \neq [\hat{\nabla}_B(p\eta) : L(w^P) \otimes p(-\alpha)] = [\hat{\nabla}_B(w^P \bullet 0) : \hat{L}(w^P \bullet 0 - p\beta)].$$

But $w^P \bullet 0 - p\beta$ is not a weight of $\hat{\nabla}_B(w^P \bullet 0)$, absurd. Thus we have obtained

Proposition: . *Assume $p \geq 4h - 5$.*

(i) $\mathcal{L}_{G/P}((w^P)^{-1} \bullet (w^P \bullet 0)^1) \otimes L((w^P \bullet 0)^0)$ appears as a direct summand of $F_* \mathcal{O}_{G/P}$.

(ii) $L((w^P \bullet 0)^0)$ is of multiplicity 1 in a composition series of $\nabla(-p(w^P)^{-1} \bullet (w^P \bullet 0)^1)$, appearing in its head.

(1.8) **Remark:** In the case of projective space $\mathbb{P}^n = \mathrm{SL}_{n+1}/P$, regardless of characteristic $\mathcal{L}_{\mathrm{SL}_{n+1}/\mathrm{SL}_{n+1,1}P}(\pi)$ always splits by [K09]. But

$$\mathcal{L}_{\mathrm{SL}_{n+1}/P}((w^P)^{-1} \bullet (w^P \bullet 0)^1) = \mathcal{O}_{\mathbb{P}^n}(r - n - 1)$$

with $r \in \mathbb{N}$ such that $rp - (n+1) \in [0, p[$, and hence $\mathcal{O}_{\mathbb{P}^n}(-n)$ is a direct summand of $F_* \mathcal{O}_{\mathbb{P}^n}$ iff $p \geq h = n + 1$ [HKR].

We will see in (4.7) that on the quadrics also $\mathcal{L}_{\bar{\mathcal{P}}}(\pi)$ splits for $p > 2$.

2° Quadrics

From now on throughout the rest of the papaer we fix an algebraically closed field $\Bbbk$ of characteristic $p \in \mathbb{N} \setminus 2$. Let E be an (n+2)-dimensional $\Bbbk$-linear space, $n \geq 3$, and $\mathbb{P} = \mathbb{P}(E) = \mathrm{Proj}(\mathrm{S}_\Bbbk(E^*))$.

(2.1) If $n = 2m+1$ is odd, let $e_1, \dots, e_{m+1}, e_0, e_{-m-1}, \dots, e_{-1}$ be a $\Bbbk$-linear basis of E and take a quadratic form

$$q(\sum_{k=-m-1}^{m+1} x_k e_k) = x_1 x_{-1} + \cdots + x_{m+1} x_{-m-1} + x_0^2, \quad x_k \in \Bbbk.$$

If $(x^t)\mathbb{B}x = \mathbb{B}(\sum x_k e_k, \sum x_k e_k) = 2q(\sum x_k e_k)$, the associated Gram matrix is given by

$$[\![\mathbb{B}(e_i, e_j) = q(e_i + e_j) - q(e_i) - q(e_j)]\!] = \left(\begin{array}{c|c|c} & & \begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix} \\ \hline & 2 & \\ \hline \begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix} & & \end{array}\right).$$

(2.2) If $n = 2m$ is even, let $e_1, \dots, e_{m+1}, e_{-m-1}, \dots, e_{-1}$ be a $\Bbbk$-linear basis of E and take a quadratic form

$$q(\sum_{k=1}^{m+1} (x_k e_k + x_{-k} e_{-k})) = x_1 x_{-1} + \cdots + x_{m+1} x_{-m-1}, \quad x_k \in \Bbbk.$$

The associated Gram matrix is

$$[\![\mathbb{B}(e_i, e_j) = q(e_i + e_j) - q(e_i) - q(e_j)]\!] = \left(\begin{array}{ccc|ccc} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ \hline & & 1 & & & \\ & \iddots & & & & \\ 1 & & & & & \end{array}\right).$$

(2.3) In either case, the projective subvariety $\mathcal{Q}^n = \operatorname{Proj}(\mathrm{S}_{\mathbb{k}}(E^*)/(q))$ of $\mathbb{P}$ defined by q is called a quadric over $\mathbb{k}$ of dimension n. We will abbreviate $\mathcal{Q}^n$ as $\mathcal{Q}$.

3° Special orthogonal groups

(3.1) Assume $n = 2m + 1$. Let $G = \mathrm{SO}(E; \mathbb{B})$. Then

$$T = \{\operatorname{diag}(\lambda_1, \dots, \lambda_{m+1}, 1, \lambda_{m+1}^{-1}, \dots, \lambda_1^{-1}) \mid \lambda_i \in \mathbb{k}^\times\ \forall i\}$$

forms a maximal torus of G with simple coroots

$$\alpha_1^\vee = \varepsilon_1^\vee - \varepsilon_2^\vee, \dots, \alpha_m^\vee = \varepsilon_m^\vee - \varepsilon_{m+1}^\vee, \alpha_{m+1}^\vee = 2\varepsilon_{m+1}^\vee,$$

where $\varepsilon_k^\vee : \zeta \mapsto \operatorname{diag}(1, \dots, 1, \zeta, 1, \dots, 1, \zeta^{-1}, 1, \dots, 1)$ with ζ appearing at the k-th place, $k \in [1, m+1]$. If $\varepsilon_k : \operatorname{diag}(\lambda_1, \dots, \lambda_{m+1}, \lambda_0, \lambda_{-m-1}, \dots, \lambda_{-1}) \mapsto \lambda_k$, the simple roots are given by

$$\begin{array}{ccccccccc} \varepsilon_1 - \varepsilon_2 & & \varepsilon_2 - \varepsilon_3 & & & & \varepsilon_m - \varepsilon_{m+1} & & \varepsilon_{m+1} \\ \circ & \text{———} & \circ & \text{———} & \cdots & \text{———} & \circ & \Longrightarrow & \circ \\ 1 & & 2 & & & & m & & m+1 \end{array}$$

The root subgroups of G are for $i, j \in [1, m+1]$ with $i < j$

$$\begin{aligned} U_{\varepsilon_i - \varepsilon_j} &= \{1 + a(e_{ij} - e_{-j,-i}) \mid a \in \mathbb{k}\}, \\ U_{-\varepsilon_i + \varepsilon_j} &= \{1 + a(e_{ji} - e_{-i,-j}) \mid a \in \mathbb{k}\}, \\ U_{\varepsilon_i + \varepsilon_j} &= \{1 + a(e_{i,-j} - e_{j,-i}) \mid a \in \mathbb{k}\}, \\ U_{-\varepsilon_i - \varepsilon_j} &= \{1 + a(e_{-ji} - e_{-ij}) \mid a \in \mathbb{k}\}, \end{aligned}$$

and for $k \in [1, m+1]$

$$U_{\varepsilon_k} = \{1 + a(2e_{k0} - e_{0,-k}) - a^2 e_{k,-k} \mid a \in \mathbb{k}\},$$

$$U_{-\varepsilon_k} = \{1 + a(e_{0k} - 2e_{-k0}) - a^2 e_{-k,k} \mid a \in \mathbb{k}\}.$$

The fundamental weights of T, defined in $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$, are given by

$$\varpi_1 = \varepsilon_1, \quad \varpi_2 = \varepsilon_1 + \varepsilon_2, \ \dots, \ \varpi_m = \varepsilon_1 + \dots + \varepsilon_m, \quad \varpi_{m+1} = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_{m+1}).$$

Here and elsewhere we will abbreviate ϖ_{α_i} as ϖ_i. The Coxeter number is $h = n + 1$.

Let W be the Weyl group of G. Then $W \simeq \mathfrak{S}_{m+1} \ltimes (\mathbb{Z}/2\mathbb{Z})^{m+1}$ permuting the ε_i, $i \in [1, m+1]$, with signs [Bou, VI.4.5]. The simple reflections are $s_1 = (1\ 2), s_2 = (2\ 3), \ldots, s_{m-1} = (m-1\ m), s_m = (m\ m+1), s_{m+1} = \binom{m+1}{-(m+1)}$. If w_0 is the longest element of W, then $w_0 = -1$.

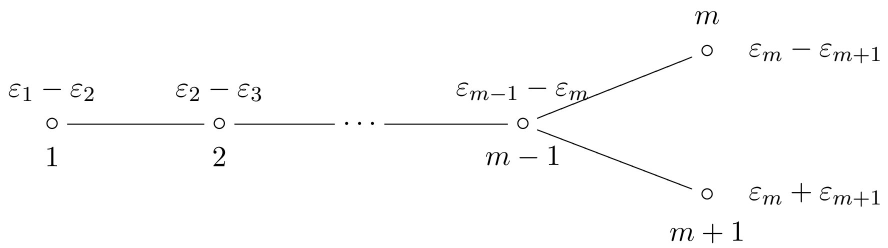
(3.2) Assume $n = 2m$. Let $G = \mathrm{SO}(E; \mathbb{B})$. Then

$$T = \{\mathrm{diag}(\lambda_1, \ldots, \lambda_{m+1}, \lambda_{m+1}^{-1}, \ldots, \lambda_1^{-1}) \mid \lambda_i \in \Bbbk^\times\ \forall i\}$$

forms a maximal torus of G with simple coroots

$$\alpha_1^\vee = \varepsilon_1^\vee - \varepsilon_2^\vee, \ldots, \alpha_{m-1}^\vee = \varepsilon_{m-1}^\vee - \varepsilon_m^\vee, \alpha_m^\vee = \varepsilon_m^\vee - \varepsilon_{m+1}^\vee, \alpha_{m+1}^\vee = \varepsilon_m^\vee + \varepsilon_{m+1}^\vee$$

with the $\varepsilon_i^\vee$ defined as in (3.1). The simple roots are given by



with the ε_i defined as in (3.1) removing λ_0. The root subgroups of G are for $i, j \in [1, m+1]$ with $i < j$

$$\begin{aligned} U_{\varepsilon_i - \varepsilon_j} &= \{1 + a(e_{ij} - e_{-j,-i}) \mid a \in \Bbbk\}, \\ U_{-\varepsilon_i + \varepsilon_j} &= \{1 + a(e_{ji} - e_{-i,-j}) \mid a \in \Bbbk\}, \\ U_{\varepsilon_i + \varepsilon_j} &= \{1 + a(e_{i,-j} - e_{j,-i}) \mid a \in \Bbbk\}, \\ U_{-\varepsilon_i - \varepsilon_j} &= \{1 + a(e_{-ji} - e_{-ij}) \mid a \in \Bbbk\}. \end{aligned}$$

The fundamental weights of T are

$$\varpi_1 = \varepsilon_1, \quad \varpi_2 = \varepsilon_1 + \varepsilon_2, \quad \ldots, \quad \varpi_{m-1} = \varepsilon_1 + \cdots + \varepsilon_{m-1},$$
$$\varpi_m = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_m - \varepsilon_{m+1}), \quad \varpi_{m+1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{m+1}).$$

The Coxeter number is $h = n$.

The Weyl group W is given by $\mathfrak{S}_{m+1} \ltimes (\mathbb{Z}/2\mathbb{Z})^m$ permuting the ε_i, $i \in [1, m+1]$, with even number of sign changes [Bou, VI.4.8]. The simple reflections are $s_1 = (1\ 2), s_2 = (2\ 3), \ldots, s_{m-1} = (m-1\ m), s_m = (m\ m+1), s_{m+1} = \begin{pmatrix} m & m+1 \\ -(m+1) & -m \end{pmatrix}$. If m is even, $w_0 = \begin{pmatrix} 1 & \ldots & m & m+1 \\ -1 & \ldots & -m & m+1 \end{pmatrix}$ on the ε_i, $i \in [1, m+1]$, and hence

$$w_0 \varpi_{m+1} = -\varpi_m, \quad w_0 \varpi_m = -\varpi_{m+1}.$$

On the other hand, if m is odd, $w_0 = -1$.

(3.3) Regardless of the parity of n, if $P = \mathrm{N}_G(\Bbbk e_{-1})$ is the stabilizer in G of the line $\Bbbk e_{-1}$, it is the standard parabolic subgroup of G whose simple roots are α_i, $i \neq 1$. One has an isomorphism of varieties $G/P \to \mathcal{Q}$ via $gP \mapsto [ge_{-1}]$. Under the imbedding $i : \mathcal{Q} \to \mathbb{P}$, define

$$\mathcal{O}_{\mathcal{Q}}(1) = i^*(\mathcal{O}_{\mathbb{P}}(1)) \simeq \mathcal{L}(\varepsilon_1) = \mathcal{L}(\omega_1). \tag{1}$$

If we identify $\mathbb{P}$ with $\mathrm{GL}(E)/\hat{P}$, $\hat{P} = \mathrm{N}_{\mathrm{GL}(E)}(\Bbbk e_{-1})$, the imbedding $i : \mathcal{Q} \to \mathbb{P}$ is compatible with the inclusion $G \hookrightarrow \mathrm{GL}(E)$, and hence G-equivariant. One has a G-equivariant short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-2) \xrightarrow{q} \mathcal{O}_{\mathbb{P}} \to i_*\mathcal{O}_{\mathcal{Q}} \to 0. \tag{2}$$

Put $\bar{S} = \coprod_{r \in \mathbb{N}} \Gamma(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(r))$. As q is irreducible in $S(E^*)$, $\bar{S} \simeq \mathrm{S}_{\Bbbk}(E^*)/(q)$, i.e., the imbedding is projectively normal. We thus obtain for each $r \in \mathbb{N}$ a G-linear short exact sequence

$$0 \to \mathrm{S}_{r-2}(E^*) \to \mathrm{S}_r(E^*) \to \bar{S}_r \to 0. \tag{3}$$

with $\bar{S}_r \simeq \nabla(r\omega_1)$, where we let $\nabla = \Gamma(G/B, \mathcal{L}(?)) = \mathrm{ind}_B^G$ denote the induction functor from the category of B-modules to the category of G-modules. By the existence of nondegenerate G-equivariant bilinear form $\mathbb{B}$ on E, $E^* \simeq E$ as G-modules, and hence (3) also reads

$$0 \to \mathrm{S}_{r-2}(E) \to \mathrm{S}_r(E) \to \nabla(r\omega_1) \to 0. \tag{4}$$

(3.4) The Weyl group W_P of P is given by

$$W_P = \begin{cases} \mathfrak{S}_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m & \text{if } n = 2m+1, \\ \mathfrak{S}_m \ltimes (\mathbb{Z}/2\mathbb{Z})^{m-1} & \text{if } n = 2m \end{cases}$$

under the identification of W with $\mathfrak{S}_{m+1} \ltimes (\mathbb{Z}/2\mathbb{Z})^{m+1}$ for $n = 2m+1$ in (3.1) and with $\mathfrak{S}_{m+1} \ltimes (\mathbb{Z}/2\mathbb{Z})^m$ for $n = 2m$ in (3.2). The longest element w_P of W_P is given by

$$w_P = \begin{cases} \begin{pmatrix} 1 & 2 & \dots & m+1 \\ 1 & -2 & \dots & -(m+1) \end{pmatrix} & \text{if } n{=}2m{+}1, \text{ or } n{=}2m \text{ with } m \text{ even}, \\ \begin{pmatrix} 1 & 2 & \dots & m & m+1 \\ 1 & -2 & \dots & -m & m+1 \end{pmatrix} & \text{if } n{=}2m \text{ with } m \text{ odd} \end{cases}$$

on the ε_i, $i \in [1, m+1]$, and hence

$$w_P\varpi_{m+1} = \begin{cases} \varpi_1 - \varpi_{m+1} & \text{if } n = 2m+1, \text{ or } n = 2m \text{ with } m \text{ even}, \\ \varpi_1 - \varpi_m & \text{if } n = 2m \text{ with } m \text{ odd} \end{cases}$$

while

$$w_P\varpi_m = \begin{cases} \varpi_1 - \varpi_m & \text{if } n = 2m \text{ with } m \text{ even}, \\ \varpi_1 - \varpi_{m+1} & \text{if } n = 2m \text{ with } m \text{ odd}. \end{cases}$$

Explicitly,

$$W^P = \begin{cases} \{e, s_1, s_2s_1, \dots, s_{m+1}s_m \dots s_2s_1, s_ms_{m+1}s_m \dots s_2s_1, \\ \qquad s_{m-1}s_ms_{m+1}s_m \dots s_2s_1, \dots, s_2 \dots s_{m-1}s_ms_{m+1}s_m \dots s_2s_1, \\ \qquad s_1s_2 \dots s_{m-1}s_ms_{m+1}s_m \dots s_2s_1\} \\ \qquad \text{if } n = 2m+1, \\ \{e, s_1, s_2s_1, \dots, s_ms_{m-1} \dots s_1, s_{m+1}s_{m-1} \dots s_1, \\ \qquad s_{m+1}s_ms_{m-1} \dots s_1 = s_ms_{m+1}s_{m-1} \dots s_1, s_{m-1}s_{m+1}s_m \dots s_1, \\ \qquad s_{m-2}s_{m-1}s_{m+1}s_m \dots s_1, \dots, s_1 \dots s_{m-1}s_{m+1}s_m \dots s_1\} \\ \qquad \text{if } n = 2m, \end{cases}$$

all of which are in reduced expression. Thus, if $n = 2m + 1$, letting $\varpi_0 = 0$, one has for each $w \in W^P$

$$w \bullet 0 = \begin{cases} -(\ell(w)+1)\varpi_{\ell(w)} + \ell(w)\varpi_{\ell(w)+1} & \text{if } \ell(w) \in [0, m-1], \\ -(m+1)\varpi_m + 2m\varpi_{m+1} & \text{if } \ell(w) = m, \\ m\varpi_m - 2(m+1)\varpi_{m+1} & \text{if } \ell(w) = m+1, \\ (\ell(w)-1)\varpi_{2m+1-\ell(w)} - \ell(w)\varpi_{2m+2-\ell(w)} & \text{if } \ell(w) \in [m+2, 2m], \\ -n\varpi_1 & \text{if } \ell(w) = n = 2m+1, \end{cases}$$

where $\ell(w) = m$ (resp. $m+1$) iff $w = s_ms_{m-1} \dots s_1$ (resp. $s_{m+1}s_ms_{m-1} \dots s_1$), and $\ell(w) = n$ iff $w = w^P = s_1 \dots s_ms_{m+1}s_ms_{m-1} \dots s_1$.

If $n = 2m$, letting $\varpi_0 = 0$, 0ne has for each $w \in W^P$

$$w \bullet 0 = \begin{cases} -(\ell(w)+1)\varpi_{\ell(w)} + \ell(w)\varpi_{\ell(w)+1} & \text{if } \ell(w) \in [0, m-2], \\ -m\varpi_{m-1} + (m-1)(\varpi_m + \varpi_{m+1}) & \text{if } \ell(w) = m-1, \\ -(m+1)\varpi_m + (m-1)\varpi_{m+1} & \text{if } w = s_ms_{m-1} \dots s_1, \\ (m-1)\varpi_m - (m+1)\varpi_{m+1} & \text{if } w = s_{m+1}s_{m-1} \dots s_1, \\ m\varpi_{m-1} - (m+1)(\varpi_m + \varpi_{m+1}) & \text{if } \ell(w) = m+1, \\ (\ell(w)-1)\varpi_{2m-\ell(w)} - \ell(w)\varpi_{2m+1-\ell(w)} & \text{if } \ell(w) \in [m+2, 2m-1], \\ -n\varpi_1 & \text{if } \ell(w) = n = 2m, \end{cases}$$

where

$\ell(w) = m-1$(resp. $m+1$) iff $w = s_{m-1}s_{m-2} \dots s_1$(resp. $s_{m+1}s_ms_{m-1} \dots s_1$),

and $\ell(w) = n$ iff $w = w^P = s_1 \dots s_{m-1}s_{m+1}s_ms_{m-1} \dots s_1$.

(3.5) Regardless of the parity of n, $(\sum_{i=1}^{m+1} \mathbb{Z}\varpi_i : \Lambda) = 2$. In order to have $\varpi_{m+1} \in \Lambda$ in the case n is odd or ϖ_{m+1} and $\varpi_m \in \Lambda$ in the case n is even, we have to go up to the simply connected covering group $G' = \mathrm{Spin}(E)$ [FH, p.308], which is the reduced Clifford group Γ_0^+ in [Ch, p.116]. By [J, II.1.17] there is a commutative diagram

$$\begin{tikzcd} G' \arrow[r] \arrow[d] & G. \\ G'/(\cap_{\lambda\in\Lambda}\ker\lambda) \arrow[ur, dotted, "\sim"'] & \end{tikzcd}$$

Thus, if B' is the Borel subgroup of G' covering B, there is for each $\mu \in \Lambda$ an isomorphism of G'-modules [J, I.6.11]

$$\mathrm{ind}_{B'}^{G'}\mu \simeq \mathrm{ind}_{B'/(\cap_{\lambda\in\Lambda}\ker\lambda)}^{G'/(\cap_{\lambda\in\Lambda}\ker\lambda)}\mu \simeq \mathrm{ind}_B^G\mu.$$

Whenever is necessary in the following, we will work with G' instead of G without mention.

(3.6) Assume finally that $p \geq h$. If $n = 2m+1$, for each $w \in W^P$

$$w^{-1} \bullet (w \bullet 0)^1 = \begin{cases} -\ell(w)\varpi_1 & \text{if } \ell(w) \in [0,m], \\ -(m+1)\varpi_1 + \varpi_{m+1} & \text{if } \ell(w) = m+1, \\ (1-\ell(w))\varpi_1 & \text{if } \ell(w) \in [m+2,n]. \end{cases}$$

If $n = 2m$, for each $w \in W^P$

$$w^{-1} \bullet (w \bullet 0)^1 = \begin{cases} -\ell(w)\varpi_1 & \text{if } \ell(w) \in [0,m-1], \\ -m\varpi_1 + \varpi_{m+1} & \text{if } w = s_m s_{m-1}\dots s_1, \\ -m\varpi_1 + \varpi_m & \text{if } w = s_{m+1}s_{m-1}\dots s_1, \\ (1-\ell(w))\varpi_1 & \text{if } \ell(w) \in [m+1,n]. \end{cases}$$

(3.7) Now, regardless of the characteristic, set $\mathcal{E}_w = \mathcal{L}(\nabla^P(w^{-1} \bullet (w \bullet 0)^1))$ $\forall w \in W^P$ with $w^{-1} \bullet (w \bullet 0)^1$ as given in (3.6). We can now state the main theorem.

Theorem: . *Assume* $\Bbbk$ *is a field of characteristic* $\neq 2$. *The locally free sheaves* $\mathcal{E}_w$, $w \in W^P$, *read, if* $n = 2m+1$,

$$\mathcal{E}_w = \begin{cases} \mathcal{O}_{\mathcal{Q}}(-\ell(w)) & \text{if } \ell(w) \in [0,m], \\ \mathcal{L}_{\mathcal{Q}}(\nabla^P(-(m+1)\varpi_1 + \varpi_{m+1})) & \text{if } \ell(w) = m+1, \\ \mathcal{O}_{\mathcal{Q}}(1-\ell(w)) & \text{if } \ell(w) \in [m+2,n], \end{cases}$$

and if $n = 2m$,

$$\mathcal{E}_w = \begin{cases} \mathcal{O}_{\mathcal{Q}}(-\ell(w)) & \text{if } \ell(w) \in [0,m-1], \\ \mathcal{L}_{\mathcal{Q}}(\nabla^P(-m\varpi_1 + \varpi_{m+1})) & \text{if } w = s_m s_{m-1}\dots s_1, \\ \mathcal{L}_{\mathcal{Q}}(\nabla^P(-m\varpi_1 + \varpi_m)) & \text{if } w = s_{m+1}s_{m-1}\dots s_1, \\ \mathcal{O}_{\mathcal{Q}}(1-\ell(w)) & \text{if } \ell(w) \in [m+1,n]. \end{cases}$$

They are all defined over $\mathbb{Z}$, and generate the bounded derived category $\mathrm{D}^b(\mathrm{coh}(\mathcal{Q}))$ *of coherent sheaves over the quadric* $\mathcal{Q} = G/P$ *such that*

(i) $\forall w \in W^P$, $\mathbf{Mod}_{\mathcal{Q}}(\mathcal{E}_w, \mathcal{E}_w) \simeq \Bbbk$,

(ii) $\forall i > 0$, $\forall x, y \in W^P$, $\mathrm{Ext}^i_{\mathcal{Q}}(\mathcal{E}_x, \mathcal{E}_y) = 0$,

(iii) $\forall x, y \in W^P$, $\mathbf{Mod}_{\mathcal{Q}}(\mathcal{E}_x, \mathcal{E}_y) \neq 0$ *iff* $x \geq y$.

(3.8) **Remarks:** (i) In characteristic 0, the $\mathcal{E}_w$, $w \in W^P$, coincide with Kapranov's sheaves [Kap].

(ii) The $\mathcal{E}_w$, $w \in W^P$, exhaust the pairwise nonisomorphic indecomposable direct summands of $F_*\mathcal{O}_{\mathcal{Q}}$ iff $p \geq h$, cf. [La].

(iii) It is important to define the $\mathcal{E}_w$, $w \in W^P$, with $w^{-1} \bullet (w \bullet 0)^1$ for $p \geq h$. For smaller p some of $w^{-1} \bullet (w \bullet 0)^1$, $w \in W^P$, coincide.

If $p = n = 2m+1$, $(s_{m+1}s_m s_{m-1} \dots s_1 \bullet 0)^1 = -2\varpi_{m+1}$ instead of $-\varpi_{m+1}$, and hence $(s_{m+1}s_m s_{m-1} \dots s_1)^{-1} \bullet (s_{m+1}s_m s_{m-1} \dots s_1 \bullet 0)^1 = -m\varpi_1$, not producing the spinor bundle.

If $p < h$, typically, regardless of the parity of n, $(w^P \bullet 0)^1 = -r\varpi_1$ with $r = \min\{i \in \mathbb{N} \,|\, pi \geq n\}$ which is ≥ 2, and hence $(w^P)^{-1} \bullet (w^P \bullet 0)^1 = (r-n)\varpi_1$.

(iv) In general, taking $\mathcal{L}_{G/P}(\nabla^P(w^{-1} \bullet (w \bullet 0)^1))$ for $\mathcal{E}_w$, $w \in W^P$, does not work to form a Karoubian complete exceptional collection if $p \geq h$. Rather, taking various pieces from $\mathrm{soc}^1_{\ell(w)+1,w}$ seems necessary [KY]; recall from (1.6) that $L(w) \otimes (\mathrm{soc}^1_{\ell(w)+1,w})^{[1]}$ is the $L(w)$-isotypic component of the $\ell(w)+1$-st subquotient of the socle series of $\hat{\nabla}_P(\Bbbk)$. Assume G is in type G_2 and let α_1 and α_2 be the simple roots with α_1 short. Let P be a parabolic subgroup of G with $R^+_L = \{\alpha_1\}$. Then $s_1s_2 \bullet 0 = -5\varpi_1 + 2\varpi_2$ and $s_1s_2s_1s_2 \bullet 0 = -6\varpi_1 + 2\varpi_2$, and hence $(s_1s_2)^{-1} \bullet (s_1s_2 \bullet 0)^1 = 2\varpi_1 - 2\varpi_2$, $(s_1s_2s_1s_2)^{-1} \bullet (s_1s_2s_1s_2 \bullet 0)^1 = 2\varpi_1 - 3\varpi_2$. We have 2 nonsplit short exact sequences of P-modules

$$0 \to \nabla^P(\varpi_1 - 2\varpi_2) \to \mathrm{soc}^1_{3,s_1s_2} \to \nabla^P(2\varpi_1 - 2\varpi_2) \to 0$$

and

$$0 \to \nabla^P(2\varpi_1 - 3\varpi_2) \to \mathrm{soc}^1_{5,s_1s_2s_1s_2} \to \nabla^P(\varpi_1 - 2\varpi_2) \to 0.$$

Our exceptional collection in [KY] has $\mathcal{E}_{s_1s_2} = \mathcal{L}_{G/P}(\mathrm{soc}^1_{3,s_1s_2})$ and $\mathcal{E}_{s_1s_2s_1s_2} = \mathcal{L}_{G/P}(\mathrm{soc}^1_{5,s_1s_2s_1s_2})$, rather than $\mathcal{E}_{s_1s_2} = \mathcal{L}_{G/P}(\nabla^P(2\varpi_1 - 2\varpi_2))$ and $\mathcal{E}_{s_1s_2s_1s_2} = \mathcal{L}_{G/P}(\nabla^P(2\varpi_1 - 3\varpi_2))$. Indeed, neither $\mathcal{E}_{s_1s_2} = \mathcal{L}_{G/P}(\nabla^P(2\varpi_1 - 2\varpi_2))$ nor

$\mathcal{E}_{s_1s_2s_1s_2} = \mathcal{L}_{G/P}(\nabla^P(2\varpi_1 - 3\varpi_2))$ is exceptional: $\forall i \in \mathbb{N}$,

$$\begin{aligned}
&\mathrm{Ext}^i_{G/P}(\mathcal{L}_{G/P}(\nabla^P(2\varpi_1 - 2\varpi_2)), \mathcal{L}_{G/P}(\nabla^P(2\varpi_1 - 2\varpi_2))) \\
&\quad \simeq \mathrm{Ext}^i_{G/P}(\mathcal{L}_{G/P}(\nabla^P(2\varpi_1 - 3\varpi_2)), \mathcal{L}_{G/P}(\nabla^P(2\varpi_1 - 3\varpi_2))) \\
&\quad \simeq \mathrm{Ext}^i_{G/P}(\mathcal{L}_{G/P}(\nabla^P(2\varpi_1)), \mathcal{L}_{G/P}(\nabla^P(2\varpi_1))) \\
&\quad \simeq \mathrm{H}^i(G/P, \mathcal{L}_{G/P}(\nabla^P(2\varpi_1)^* \otimes \nabla^P(2\varpi_1))) \\
&\quad \simeq \mathrm{H}^i(G/P, \mathcal{L}_{G/P}(\nabla^P(2\varpi_1 - 2\varpi_2) \otimes \nabla^P(2\varpi_1))) \\
&\quad \simeq \begin{cases} \Bbbk & \text{if } i = 0 \\ \nabla(\varpi_1) & \text{if } i = 1 \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

Note that $\mathrm{soc}^1_{3,s_1s_2}$ is a P-module generated by a vector of weight $(s_1s_2)^{-1} \bullet (s_1s_2 \bullet 0)^1 = 2\varpi_1 - 2\varpi_2$ while $\mathrm{soc}^1_{5,s_1s_2s_1s_2}$ is a P-module generated by a vector of weight $\varpi_1 - 2\varpi_2$.

As we skipped details in [KY], let us exhibit here an argument to show that $\mathcal{E}_{s_1s_2} = \mathcal{L}_{G/P}(\mathrm{soc}^1_{3,s_1s_2})$ is exceptional. Put $\mathcal{P} = G/P$ and $\mathcal{B} = G/B$. We first note that $\mathrm{soc}^1_{3,s_1s_2}$ may by the unicity of the extension be written also as $(-\varpi_2) \otimes \ker(\Delta(\varpi_1) \twoheadrightarrow \Delta^P(\varpi_1))$ with $\Delta(\varpi_1)$ (resp. $\Delta^P(\varpi_1)$) denoting the Weyl module of highest weight ϖ_1 for G (resp. P, i.e., for the Levi of P). This expression yields a long exact sequence

$$\begin{aligned}
&0 \to \mathbf{Mod}_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}((-\varpi_2) \otimes \Delta^P(\varpi_1)), \mathcal{E}_{s_1s_2}) \to \\
&\qquad \mathbf{Mod}_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}((-\varpi_2) \otimes \Delta(\varpi_1)), \mathcal{E}_{s_1s_2}) \to \mathbf{Mod}_{\mathcal{P}}(\mathcal{E}_{s_1s_2}, \mathcal{E}_{s_1s_2}) \to \dots
\end{aligned}$$

with $\mathrm{Ext}^\bullet_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}((-\varpi_2) \otimes \Delta(\varpi_1)), \mathcal{E}_{s_1s_2}) \simeq \nabla(\varpi_1) \otimes \mathrm{Ext}^\bullet_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}(-\varpi_2), \mathcal{E}_{s_1s_2}) \simeq \nabla(\varpi_1) \otimes \mathrm{H}^\bullet(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\varpi_2) \otimes_{\mathcal{P}} \mathcal{E}_{s_1s_2}) = 0$ as both $\mathrm{H}^\bullet(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\varpi_2 \otimes \nabla^P(\varpi_1 - 2\varpi_2)) \simeq \mathrm{H}^\bullet(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\varpi_1 - \varpi_2))$ and $\mathrm{H}^\bullet(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\varpi_2 \otimes \nabla^P(2\varpi_1 - 2\varpi_2)) \simeq \mathrm{H}^\bullet(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(2\varpi_1 - \varpi_2))$ vanish. It follows $\forall i \in \mathbb{N}$ that $\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}_{s_1s_2}, \mathcal{E}_{s_1s_2}) \simeq \mathrm{Ext}^{i+1}_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}((-\varpi_2) \otimes \Delta^P(\varpi_1)), \mathcal{E}_{s_1s_2})$. On the other hand, we have another long exact sequence

$$\begin{aligned}
&0 \to \mathbf{Mod}_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}((-\varpi_2) \otimes \Delta^P(\varpi_1)), \mathcal{E}_{s_1s_2}) \to \\
&\qquad \mathbf{Mod}_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}((-\varpi_2) \otimes \Delta^P(\varpi_1)), \mathcal{L}_{\mathcal{P}}((-\varpi_2) \otimes \Delta(\varpi_1))) \to \\
&\qquad\quad \mathbf{Mod}_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}((-\varpi_2) \otimes \Delta^P(\varpi_1)), \mathcal{L}_{\mathcal{P}}((-\varpi_2) \otimes \Delta^{\alpha_1}(\varpi_1))) \to \dots
\end{aligned}$$

with

$$\begin{aligned}&\mathrm{Ext}^\bullet_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}((-\varpi_2)\otimes\Delta^{P1}(\varpi_1)),\mathcal{L}_{\mathcal{P}}((-\varpi_2)\otimes\Delta(\varpi_1)))\\ &\qquad\simeq\Delta(\varpi_1)\otimes\mathrm{H}^\bullet(\mathcal{P},\mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_1-\varpi_2)))=0.\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}_{s_1s_2},\mathcal{E}_{s_1s_2})&\simeq\mathrm{Ext}^{i+1}_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}((-\varpi_2)\otimes\Delta^P(\varpi_1)),\mathcal{E}_{s_1s_2})\\ &\simeq\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}((-\varpi_2)\otimes\Delta(\varpi_1)),\mathcal{L}_{\mathcal{P}}((-\varpi_2)\otimes\Delta^P(\varpi_1)))\\ &\simeq\mathrm{H}^i(\mathcal{P},\mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_1-\varpi_2)\otimes\nabla^P(\varpi_1)))\\ &\simeq\mathrm{H}^i(\mathcal{P},\mathcal{L}_{\mathcal{P}}(\Bbbk))\\ &\quad\text{as } \nabla^P(\varpi_1-\varpi_2)\otimes\nabla^P(\varpi_1) \text{ is an extension of } \Bbbk \text{ by } \nabla^P(2\varpi_1-\varpi_2)\\ &\simeq\delta_{i0}\Bbbk.\end{aligned}$$

4° Extensions

In this section we let $\Bbbk$ denote a field of characteristic $\neq 2$ and compute the extensions among the $\mathcal{E}_w$, $w\in W^P$, using representation theory. As the computations are similar for n odd/even, we will only write down the proofs for n odd. As will be clear from the arguments, the fact that ϖ_{m+1} (resp. ϖ_m and ϖ_{m+1} in case n is even) is minuscule plays a key role. For each $\lambda\in\Lambda^+$ let $\Delta(\lambda)=\nabla(-w_0\lambda)^*$ denote the Weyl module for G of highest weight λ.

(4.1) **Proposition:** *(i) Assume $n=2m+1$. One has $\nabla(\varpi_{m+1})$ irreducible with $\nabla(\varpi_{m+1})^*\simeq\nabla(\varpi_{m+1})$ and $\dim\nabla(\varpi_{m+1})=2^{m+1}=2\dim\nabla^P(\varpi_{m+1})$. There is an isomorphism of P-modules*

$$\nabla^P(\varpi_{m+1})^*\simeq\nabla^P(\varpi_{m+1})\otimes(-\varpi_1).$$

One has

$$P\mathbf{Mod}(\nabla(\varpi_{m+1}),\nabla^P(\varpi_{m+1}))\simeq\Bbbk\simeq P\mathbf{Mod}(\nabla^P(\varpi_{m+1})\otimes(-\varpi_1),\nabla(\varpi_{m+1})),$$

whose nonzero morphisms constitute a nonsplit short exact sequence of P-modules

$$0\to\nabla^P(\varpi_{m+1})\otimes(-\varpi_1)\to\nabla(\varpi_{m+1})\to\nabla^P(\varpi_{m+1})\to 0,$$

inducing a long exact sequence of P-modules

$$\begin{aligned}\cdots\to\nabla(\varpi_{m+1})\otimes(-2\varpi_1)\to\nabla(\varpi_{m+1})\otimes(-\varpi_1)\to&\\ \nabla(\varpi_{m+1})\to\nabla^P(\varpi_{m+1})\to 0.&\end{aligned}$$

(ii) Assume $n = 2m$. Both $\nabla(\varpi_{m+1})$ and $\nabla(\varpi_m)$ are irreducible with

$$\nabla(\varpi_{m+1})^* \simeq \begin{cases} \nabla(\varpi_m) & \text{if } m \text{ is even} \\ \nabla(\varpi_{m+1}) & \text{if } m \text{ is odd} \end{cases}$$

and

$$\nabla(\varpi_m)^* \simeq \begin{cases} \nabla(\varpi_{m+1}) & \text{if } m \text{ is even} \\ \nabla(\varpi_m) & \text{if } m \text{ is odd,} \end{cases}$$

$\dim \nabla(\varpi_{m+1}) = \dim \nabla(\varpi_m) = 2^m = 2 \dim \nabla^P(\varpi_{m+1}) = 2 \dim \nabla^P(\varpi_m)$. *There are isomorphisms of P-modules*

$$\nabla^P(\varpi_m)^* \simeq \begin{cases} \nabla^P(\varpi_m) \otimes (-\varpi_1) & \text{if } m \text{ is even} \\ \nabla^P(\varpi_{m+1}) \otimes (-\varpi_1) & \text{if } m \text{ is odd,} \end{cases}$$

$$\nabla^P(\varpi_{m+1})^* \simeq \begin{cases} \nabla^P(\varpi_{m+1}) \otimes (-\varpi_1) & \text{if } m \text{ is even} \\ \nabla^P(\varpi_m) \otimes (-\varpi_1) & \text{if } m \text{ is odd.} \end{cases}$$

One has

$$P\mathbf{Mod}(\nabla(\varpi_{m+1}), \nabla^P(\varpi_{m+1})) \simeq \Bbbk \simeq P\mathbf{Mod}(\nabla^P(\varpi_m) \otimes (-\varpi_1), \nabla(\varpi_{m+1})),$$

whose nonzero morphisms constitute a nonsplit short exact sequence of P-modules

$$0 \to \nabla^P(\varpi_m) \otimes (-\varpi_1) \to \nabla(\varpi_{m+1}) \to \nabla^P(\varpi_{m+1}) \to 0.$$

Likewise

$$P\mathbf{Mod}(\nabla(\varpi_m), \nabla^P(\varpi_m)) \simeq \Bbbk \simeq P\mathbf{Mod}(\nabla^P(\varpi_{m+1}) \otimes (-\varpi_1), \nabla(\varpi_m)),$$

whose nonzero morphisms constitute a nonsplit short exact sequence of P-modules

$$0 \to \nabla^P(\varpi_{m+1}) \otimes (-\varpi_1) \to \nabla(\varpi_m) \to \nabla^P(\varpi_m) \to 0.$$

Combining the 2 SES's yields 2 long exact sequences of P-modules

$$\begin{aligned} \cdots \to \nabla(\varpi_{m+1}) \otimes (-4\varpi_1) \to \nabla(\varpi_m) \otimes (-3\varpi_1) \to \nabla(\varpi_{m+1}) \otimes (-2\varpi_1) \to \\ \nabla(\varpi_m) \otimes (-\varpi_1) \to \nabla(\varpi_{m+1}) \to \nabla^P(\varpi_{m+1}) \to 0. \end{aligned}$$

and

$$\begin{aligned} \cdots \to \nabla(\varpi_m) \otimes (-4\varpi_1) \to \nabla(\varpi_{m+1}) \otimes (-3\varpi_1) \to \nabla(\varpi_m) \otimes (-2\varpi_1) \to \\ \nabla(\varpi_{m+1}) \otimes (-\varpi_1) \to \nabla(\varpi_m) \to \nabla^P(\varpi_m) \to 0. \end{aligned}$$

Proof: (i) As ϖ_{m+1} is minuscule, $\nabla(\varpi_{m+1})$ (resp. $\nabla^P(\varpi_{m+1})$) is irreducible for G (resp. P). Then $\nabla(\varpi_{m+1})^* = L(\varpi_{m+1})^* = L(-w_0\varpi_{m+1}) = L(\varpi_{m+1}) = \nabla(\varpi_{m+1})$. Likewise

$$\nabla^P(\varpi_{m+1})^* \simeq \nabla^P(-w_P\varpi_{m+1}) \simeq \nabla^P(-\varpi_1 + \varpi_{m+1}) \simeq \nabla^P(\varpi_{m+1}) \otimes (-\varpi_1). \tag{1}$$

One has

$$P\mathbf{Mod}(\nabla(\varpi_{m+1}), \nabla^P(\varpi_{m+1})) \simeq B\mathbf{Mod}(\nabla(\varpi_{m+1}), \varpi_{m+1}) \simeq \Bbbk$$

while

$$\begin{aligned} &P\mathbf{Mod}(\nabla^P(\varpi_{m+1}) \otimes (-\varpi_1), \nabla(\varpi_{m+1})) \\ &\simeq P\mathbf{Mod}(\nabla(\varpi_{m+1})^*, \varpi_1 \otimes \nabla^P(\varpi_{m+1})^*) \\ &\simeq P\mathbf{Mod}(\Delta(-w_0\varpi_{m+1}), \varpi_1 \otimes \nabla^P(-w_P\varpi_{m+1})) \\ &= P\mathbf{Mod}(\Delta(\varpi_{m+1}), \varpi_1 \otimes \nabla^P(-\varpi_1 + \varpi_{m+1})) \quad \text{by (1)} \\ &\simeq P\mathbf{Mod}(\Delta(\varpi_{m+1}), \nabla^P(\varpi_{m+1})) \simeq G\mathbf{Mod}(\Delta(\varpi_{m+1}), \nabla(\varpi_{m+1})) \\ &\simeq \Bbbk \quad \text{by [J, II.4.16].} \end{aligned}$$

As ϖ_{m+1} is minuscule again,

$$\begin{aligned} \dim \nabla^P(\varpi_{m+1}) &= |W_P\varpi_{m+1}| = |W_P/\mathrm{C}_{W_P}(\varpi_{m+1})| \\ &= |(\mathfrak{S}_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m)/\mathrm{C}_{W_P}(\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{m+1}))| \\ &= |(\mathfrak{S}_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m)/\mathfrak{S}_m| \\ &= |(\mathbb{Z}/2\mathbb{Z})^m| = 2^m, \end{aligned}$$

and likewise $\dim \nabla(\varpi_{m+1}) = 2^{m+1}$. Then by the irreducibility of $\nabla^P(\varpi_{m+1})$ and by dimension one obtains a short exact sequence of P-modules

$$0 \to \nabla^P(\varpi_{m+1}) \otimes (-\varpi_1) \to \nabla(\varpi_{m+1}) \to \nabla^P(\varpi_{m+1}) \to 0, \qquad (2)$$

tensoring which with $r\varpi_1$, $r \in \mathbb{Z}$, yields another short exact sequence

$$0 \to \nabla^P(\varpi_{m+1}) \otimes (r-1)\varpi_1 \to \nabla(\varpi_{m+1}) \otimes r\varpi_1 \to \nabla^P(\varpi_{m+1}) \otimes r\varpi_1 \to 0.$$

Combining these exact sequences one obtains the asserted long exact sequence.

As $P\mathbf{Mod}(\nabla(\varpi_{m+1}), \nabla^P(\varpi_{m+1}) \otimes (-\varpi_1)) \simeq P\mathbf{Mod}(\nabla(\varpi_{m+1}), \nabla^P(\varpi_{m+1} - \varpi_1)) \simeq B\mathbf{Mod}(\nabla(\varpi_{m+1}), \varpi_{m+1} - \varpi_1) \simeq G\mathbf{Mod}(\nabla(\varpi_{m+1}), \nabla(\varpi_{m+1} - \varpi_1)) = 0$, the sequence (2) is nonsplit.

(ii) Note that ϖ_{m+1} and ϖ_m are both minuscule.

(4.2) Sheafifying the short exact sequences of P-modules in (4.1) yields

Proposition: . *(i) If $n = 2m+1$, there is a short exact sequence of G-equivariant $\mathcal{O}_\mathcal{Q}$-modules*

$$0 \to \mathcal{L}(\nabla^P(\varpi_{m+1}))(-1) \to \mathcal{O}_\mathcal{Q} \otimes \nabla(\varpi_{m+1}) \to \mathcal{L}(\nabla^P(\varpi_{m+1})) \to 0.$$

(ii) If $n = 2m$, there are short exact sequences of G-equivariant $\mathcal{O}_\mathcal{Q}$-modules

$$0 \to \mathcal{L}(\nabla^P(\varpi_m))(-1) \to \mathcal{O}_\mathcal{Q} \otimes \nabla(\varpi_{m+1}) \to \mathcal{L}(\nabla^P(\varpi_{m+1})) \to 0$$

and

$$0 \to \mathcal{L}(\nabla^P(\varpi_{m+1}))(-1) \to \mathcal{O}_{\mathcal{Q}} \otimes \nabla(\varpi_m) \to \mathcal{L}(\nabla^P(\varpi_m)) \to 0.$$

(4.3) **Proposition:** *Let $r, j \in \mathbb{N}$.*

(i) $\dim \nabla(r\varpi_1) = \dfrac{n+2r}{n}\dbinom{n+r-1}{r}$.

(ii) There is an isomorphism of G-modules $\mathrm{H}^j(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(r)) \simeq \delta_{j0}\nabla(r\varpi_1)$.

(iii) If $r > 0$, there are isomorphism of G-modules

$$\mathrm{H}^j(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(-r)) \simeq \delta_{jn}\nabla((r-n)\varpi_1)^* \simeq \delta_{jn}\Delta((r-n)\varpi_1).$$

Proof: (i) From the exact sequence (3.3.4)

$$\begin{aligned}\dim \nabla(r\varpi_1) &= \begin{cases} \binom{n+2+r-1}{r} - \binom{n+2+r-2-1}{r-2} & \text{if } r \geq 2 \\ \binom{n+2+r-1}{r} & \text{if } r = 0 \text{ or } 1\end{cases} \\ &= \binom{n+r-1}{r}\frac{n+2r}{n}.\end{aligned}$$

(ii) follows from Kempf's vanishing [J, II.4.5].

(iii) From (3.3.2) one obtains for each $k \in \mathbb{Z}$ an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(k-2) \to \mathcal{O}_{\mathbb{P}}(k) \to (i_*\mathcal{O}_{\mathcal{Q}})(k) \to 0.$$

with $(i_*\mathcal{O}_{\mathcal{Q}})(k) \simeq i_*(\mathcal{O}_{\mathcal{Q}}(k))$ by the projection formula. As $\mathbb{P} = \mathbb{P}^{n+1}$, $\mathrm{H}^j(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)) = 0 \ \forall j \in]0, n+1[$. Then, as i is a closed immersion, $\forall j \in]0, n[$,

$$\mathrm{H}^j(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(k)) \simeq \mathrm{H}^j(\mathbb{P}, i_*(\mathcal{O}_{\mathcal{Q}}(k))) = 0. \tag{1}$$

Also $\forall j \in \mathbb{N}$,

$$\begin{aligned}\mathrm{H}^j(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(-r)) &\simeq \delta_{jn}\mathrm{H}^j(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(-r)) \text{ by (1) and the Grothendieck vanishing} \\ &= \delta_{jn}\mathrm{H}^0(\mathcal{Q}, \mathcal{L}(r\varpi_1 - 2\rho_P))^* \quad \text{by the Serre duality [J, II.4.2]} \\ &\simeq \delta_{jn}\nabla(r\varpi_1 - 2\rho_P)^* \simeq \delta_{jn}\Delta(-w_0(r\varpi_1 - 2\rho_P)).\end{aligned}$$

As $2\rho_P = n\varpi_1$, the assertion follows.

(4.4) **Proposition:** *Let $j, r \in \mathbb{N}$ with $r > 0$.*

(i) Assume $n = 2m+1$. Then $\mathrm{H}^\bullet(\mathcal{Q}, \mathcal{L}(\nabla^P(\varpi_{m+1}))(-1)) = 0$ while there are isomorphisms of G-modules

$$\mathrm{H}^j(\mathcal{Q}, \mathcal{L}(\nabla^P(\varpi_{m+1}))(r-1)) \simeq \delta_{j0}\nabla((r-1)\varpi_1 + \varpi_{m+1})$$

with

$$\nabla((r-1)\varpi_1 + \varpi_{m+1}) \simeq \{\nabla((r-1)\varpi_1) \otimes \nabla(\varpi_{m+1})\}/\nabla((r-2)\varpi_1 + \varpi_{m+1})$$

of dimension $2^{m+1}\binom{n+r-1}{n}$, *and*

$$\mathrm{H}^j(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_{m+1}))(-r-1))\simeq\delta_{jn}\nabla((r-n)\varpi_1+\varpi_{m+1})^*.$$

(ii) Assume $n=2m$ *with* m *even. Then*

$$\mathrm{H}^\bullet(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_{m+1}))(-1))=0=\mathrm{H}^\bullet(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_m))(-1))$$

while there are isomorphisms of G*-modules*

$$\begin{aligned}\mathrm{H}^j(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_{m+1}))(r-1))&\simeq\delta_{j0}\nabla((r-1)\varpi_1+\varpi_{m+1}),\\ \mathrm{H}^j(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_m))(r-1))&\simeq\delta_{j0}\nabla((r-1)\varpi_1+\varpi_m)\end{aligned}$$

with

$$\begin{aligned}\nabla((r-1)\varpi_1+\varpi_{m+1})&\simeq\{\nabla((r-1)\varpi_1)\otimes\nabla(\varpi_{m+1})\}/\nabla((r-2)\varpi_1+\varpi_m),\\ \nabla((r-1)\varpi_1+\varpi_m)&\simeq\{\nabla((r-1)\varpi_1)\otimes\nabla(\varpi_m)\}/\nabla((r-2)\varpi_1+\varpi_{m+1})\end{aligned}$$

both of dimension $2^m\binom{n+r-1}{n}$, *and*

$$\begin{aligned}\mathrm{H}^j(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_{m+1}))(-r-1))&\simeq\delta_{jn}\nabla((r-n)\varpi_1+\varpi_{m+1})^*,\\ \mathrm{H}^j(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_m))(-r-1))&\simeq\delta_{jn}\nabla((r-n)\varpi_1+\varpi_m)^*.\end{aligned}$$

(iii) Assume $n=2m$ *with* m *odd. Then*

$$\mathrm{H}^\bullet(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_{m+1}))(-1))=0=\mathrm{H}^\bullet(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_m))(-1))$$

while there are isomorphisms of G*-modules*

$$\begin{aligned}\mathrm{H}^j(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_{m+1}))(r-1))&\simeq\delta_{j0}\nabla((r-1)\varpi_1+\varpi_{m+1}),\\ \mathrm{H}^j(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_m))(r-1))&\simeq\delta_{j0}\nabla((r-1)\varpi_1+\varpi_m)\end{aligned}$$

with

$$\begin{aligned}\nabla((r-1)\varpi_1+\varpi_m)&\simeq\{\nabla((r-1)\varpi_1)\otimes\nabla(\varpi_m)\}/\nabla((r-2)\varpi_1+\varpi_{m+1}),\\ \nabla((r-1)\varpi_1+\varpi_{m+1})&\simeq\{\nabla((r-1)\varpi_1)\otimes\nabla(\varpi_{m+1})\}/\nabla((r-2)\varpi_1+\varpi_m)\end{aligned}$$

both of dimension $2^m\binom{n+r-1}{n}$, *and*

$$\begin{aligned}\mathrm{H}^j(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_{m+1}))(-r-1))&\simeq\delta_{jn}\nabla((r-n)\varpi_1+\varpi_m)^*,\\ \mathrm{H}^j(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_m))(-r-1))&\simeq\delta_{jn}\nabla((r-n)\varpi_1+\varpi_{m+1})^*.\end{aligned}$$

Proof: (i) One has

$$\begin{aligned}\mathrm{H}^\bullet(\mathcal{Q}, \mathcal{L}(\nabla^P(\varpi_{m+1}))(-1)) &\simeq \mathrm{H}^\bullet(\mathcal{B}, \mathcal{L}(\varpi_{m+1} - \varpi_1)) \text{by the degeneracy} \\ &\qquad \text{of the spectral sequence} \\ &\qquad \mathbf{R}^i\mathrm{ind}_P^G(\mathbf{R}^j\mathrm{ind}_B^P?) \Rightarrow \mathbf{R}^{i+j}\mathrm{ind}_B^G? \\ &= 0 \quad \text{from [J, II.5.4].}\end{aligned}$$

Recall from (4.2.i) a short exact sequence of G-equivariant $\mathcal{O}_\mathcal{Q}$-modules

$$0 \to \mathcal{L}(\nabla^P(\varpi_{m+1}))(-1) \to \mathcal{O}_\mathcal{Q} \otimes \nabla(\varpi_{m+1}) \to \mathcal{L}(\nabla^P(\varpi_{m+1})) \to 0.$$

One has $\forall k \in \mathbb{Z}$ a short exact sequence of G-equivariant $\mathcal{O}_\mathcal{Q}$-modules

$$0 \to \mathcal{L}(\nabla^P(\varpi_{m+1}))(k-1) \to \mathcal{O}_\mathcal{Q}(k) \otimes \nabla(\varpi_{m+1}) \to \mathcal{L}(\nabla^P(\varpi_{m+1}))(k) \to 0. \tag{1}$$

As $\mathrm{H}^j(\mathcal{Q}, \mathcal{L}(\nabla^P(\varpi_{m+1}))(r-1)) \simeq \delta_{j0}\nabla((r-1)\varpi_1 + \varpi_{m+1})$ by Kempf, one obtains a short exact sequence of G-modules

$$\begin{aligned}0 \to \nabla((r-2)\varpi_1 + \varpi_{m+1}) \to \nabla((r-1)\varpi_1) \otimes \nabla(\varpi_{m+1}) \to \\ \nabla((r-1)\varpi_1 + \varpi_{m+1}) \to 0.\end{aligned} \tag{2}$$

We know from (4.1) that $\dim \nabla(\varpi_{m+1}) = 2^{m+1}$. Assume by induction that $\dim \nabla((r-1)\varpi_1 + \varpi_{m+1}) = 2^{m+1}\binom{n+r-1}{n}$. Then by (2)

$$\begin{aligned}&\dim\nabla(r\varpi_1 + \varpi_{m+1}) \\ &\quad = \dim(\nabla(r\varpi_1) \otimes \nabla(\varpi_{m+1})) - \dim \nabla((r-1)\varpi_1 + \varpi_{m+1}) \\ &\quad = \frac{n+2r}{n}\binom{n+r-1}{r}2^{m+1} - 2^{m+1}\binom{n+r-1}{n} \quad \text{by (4.3)} \\ &\quad = 2^{m+1}\binom{n+r}{n}.\end{aligned}$$

$\forall j < n$, the short exact sequence (1) yields by (4.3)

$$\begin{aligned}\mathrm{H}^j(\mathcal{Q}, \mathcal{L}(\nabla^P(\varpi_{m+1}))(-r-1)) &\simeq \mathrm{H}^{j-1}(\mathcal{Q}, \mathcal{L}(\nabla^P(\varpi_{m+1}))(-r)) \simeq \dots \\ &\simeq \mathrm{H}^1(\mathcal{Q}, \mathcal{L}(\nabla^P(\varpi_{m+1}))(j-r-2))\end{aligned}$$

and an exact sequence

$$\begin{aligned}\nabla(\varpi_{m+1}) \otimes \nabla((j-r-1)\varpi_1) \to \nabla((j-r-1)\varpi_1 + \varpi_{m+1})) \to \\ \mathrm{H}^1(\mathcal{Q}, \mathcal{L}(\nabla^P(\varpi_{m+1}))(j-r-2)) \to 0.\end{aligned}$$

But the first morphism is the cup product and is surjective by Donkin [J, II.4.21], and hence $\mathrm{H}^1(\mathcal{Q}, \mathcal{L}(\nabla^P(\varpi_{m+1}))(j-r-2)) = 0$. If $j = n$,

$$\begin{aligned}\mathrm{H}^n(\mathcal{Q}, &\mathcal{L}(\nabla^P(\varpi_{m+1}))(-r-1)) \simeq \mathrm{H}^n(G/P, \mathcal{L}((-r-1)\varpi_1 \otimes \nabla^P(\varpi_{m+1})))\\ &\simeq \mathrm{H}^0(G/P, \mathcal{L}((r+1)\varpi_1 \otimes \nabla^P(\varpi_{m+1}) \otimes (-\varpi_1) \otimes (-n\varpi_1)))^*\\ &\qquad\qquad\qquad\text{by the Serre duality and (4.1)}\\ &\simeq \nabla((r-n)\varpi_1 + \varpi_{m+1})^*.\end{aligned}$$

(4.5) **Lemma:** *(i) If $n = 2m+1$, $\Delta(\varpi_{m+1}) \otimes \Delta(\varpi_{m+1})$ admits a filtration of G-modules whose subquotients are $\Delta(\Bbbk)$, $\Delta(\varpi_1)$, $\Delta(\varpi_2)$, $\dots$, $\Delta(\varpi_m)$, $\Delta(2\varpi_{m+1})$, each occuring once.*

(ii) Assume $n = 2m$ with m even. $\Delta(\varpi_{m+1}) \otimes \Delta(\varpi_{m+1})$ admits a filtration of G-modules whose subquotients are $\Delta(\varpi_1)$, $\Delta(\varpi_3)$, $\dots$, $\Delta(\varpi_{m-3})$, $\Delta(\varpi_{m-1})$, $\Delta(2\varpi_{m+1})$, each occuring once. $\Delta(\varpi_{m+1}) \otimes \Delta(\varpi_m)$ admits a filtration of G-modules whose subquotients are $\Delta(\Bbbk)$, $\Delta(\varpi_2)$, $\dots$, $\Delta(\varpi_{m-4})$, $\Delta(\varpi_{m-2})$, $\Delta(\varpi_{m+1} + \varpi_m)$, each occuring once. $\Delta(\varpi_m) \otimes \Delta(\varpi_m)$ admits a filtration of G-modules whose subquotients are $\Delta(\varpi_1)$, $\Delta(\varpi_3)$, $\dots$, $\Delta(\varpi_{m-3})$, $\Delta(\varpi_{m-1})$, $\Delta(2\varpi_m)$, each occuring once.

(iii) Assume $n = 2m$ with m odd. $\Delta(\varpi_{m+1}) \otimes \Delta(\varpi_{m+1})$ admits a filtration of G-modules whose subquotients are $\Delta(\Bbbk)$, $\Delta(\varpi_2)$, $\dots$, $\Delta(\varpi_{m-3})$, $\Delta(\varpi_{m-1})$, $\Delta(2\varpi_{m+1})$, each occuring once. $\Delta(\varpi_{m+1}) \otimes \Delta(\varpi_m)$ admits a filtration of G-modules whose subquotients are $\Delta(\varpi_1)$, $\Delta(\varpi_3)$, $\dots$, $\Delta(\varpi_{m-4})$, $\Delta(\varpi_{m-2})$, $\Delta(\varpi_{m+1} + \varpi_m)$, each occuring once. $\Delta(\varpi_m) \otimes \Delta(\varpi_m)$ admits a filtration of G-modules whose subquotients are $\Delta(\Bbbk)$, $\Delta(\varpi_2)$, $\dots$, $\Delta(\varpi_{m-3})$, $\Delta(\varpi_{m-1})$, $\Delta(2\varpi_m)$, each occuring once.

Proof: By Lusztig's theory of based modules [L, 27] we may transfer to the corresponding quantum group, cf. [X], [K98]. There the assertions follow from [Kas, 4.2], [KN, 5.4, 6.4].

(4.6) **Theorem:** *Let $k \in \mathbb{N}$ and $i, j \in \mathbb{Z}$.*

(i) Assume $n = 2m + 1$. Then

$$\operatorname{Ext}^k_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_{m+1})), \mathcal{L}(\nabla^P(\varpi_{m+1}))) \simeq \delta_{k0}\Bbbk,$$

$$\operatorname{Ext}^k_{\mathcal{Q}}(\mathcal{O}_{\mathcal{Q}}(i), \mathcal{O}_{\mathcal{Q}}(j)) \simeq \begin{cases} \delta_{k0}\nabla((j-i)\varpi_1) & \text{if } j-i \geq 0 \\ \delta_{kn}\nabla((i-j-n)\varpi_1)^* & \\ \quad \simeq \delta_{kn}\Delta((i-j-n)\varpi_1) & \text{if } j-i \leq -n, \end{cases}$$

$$\operatorname{Ext}^k_{\mathcal{Q}}(\mathcal{O}_{\mathcal{Q}}(i), \mathcal{L}(\nabla^P(\varpi_{m+1}))(j)) \simeq \begin{cases} \delta_{k0}\nabla((j-i)\varpi_1 + \varpi_{m+1}) \\ \quad \text{if } j-i \geq 0, \\ \delta_{kn}\nabla((i-j-1-n)\varpi_1 + \varpi_{m+1})^* \\ \quad \text{if } j-i \leq -n-1, \\ 0 \quad \text{else}, \end{cases}$$

$$\operatorname{Ext}^k_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_{m+1}))(i), \mathcal{O}_{\mathcal{Q}}(j)) \simeq \begin{cases} \delta_{k0}\nabla((j-i-1)\varpi_1 + \varpi_{m+1}) \\ \quad \text{if } j-i \geq 1, \\ \delta_{kn}\nabla((i-j-n)\varpi_1 + \varpi_{m+1})^* \\ \quad \text{if } j-i \leq -n, \\ 0 \quad \text{else.} \end{cases}$$

(ii) Assume $n = 2m$ with m even. Then

$$\begin{aligned}\operatorname{Ext}^k_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_{m+1})), \mathcal{L}(\nabla^P(\varpi_{m+1}))) &\simeq \delta_{k0}\Bbbk \\ &\simeq \operatorname{Ext}^k_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_m)), \mathcal{L}(\nabla^P(\varpi_m))),\end{aligned}$$

$$\begin{aligned}\operatorname{Ext}^\bullet_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_{m+1})), \mathcal{L}(\nabla^P(\varpi_m))) &= 0 \\ &= \operatorname{Ext}^\bullet_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_m)), \mathcal{L}(\nabla^P(\varpi_{m+1}))),\end{aligned}$$

$$\operatorname{Ext}^k_{\mathcal{Q}}(\mathcal{O}_{\mathcal{Q}}(i), \mathcal{O}_{\mathcal{Q}}(j)) \simeq \begin{cases} \delta_{k0}\nabla((j-i)\varpi_1) & \text{if } j-i \geq 0 \\ \delta_{kn}\nabla((i-j-n)\varpi_1)^* & \\ \quad \simeq \delta_{kn}\Delta((i-j-n)\varpi_1) & \text{if } j-i \leq -n \\ 0 & \text{else}, \end{cases}$$

$$\mathrm{Ext}^k_{\mathcal{Q}}(\mathcal{O}_{\mathcal{Q}}(i), \mathcal{L}(\nabla^P(\varpi_{m+1}))(j)) \simeq \begin{cases} \delta_{k0}\nabla((j-i)\varpi_1 + \varpi_{m+1}) \\ \qquad \textit{if } j-i \geq 0, \\ \\ \delta_{kn}\nabla((i-j-1-n)\varpi_1 + \varpi_{m+1})^* \\ \qquad \textit{if } j-i \leq -n-1, \\ \\ 0 \quad \textit{else}, \end{cases}$$

$$\mathrm{Ext}^k_{\mathcal{Q}}(\mathcal{O}_{\mathcal{Q}}(i), \mathcal{L}(\nabla^P(\varpi_m))(j)) \simeq \begin{cases} \delta_{k0}\nabla((j-i)\varpi_1 + \varpi_m) \\ \qquad \textit{if } j-i \geq 0, \\ \\ \delta_{kn}\nabla((i-j-1-n)\varpi_1 + \varpi_m)^* \\ \qquad \textit{if } j-i \leq -n-1, \\ \\ 0 \quad \textit{else}, \end{cases}$$

$$\mathrm{Ext}^k_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_{m+1}))(i), \mathcal{O}_{\mathcal{Q}}(j)) \simeq \begin{cases} \delta_{k0}\nabla((j-i-1)\varpi_1 + \varpi_{m+1}) \\ \qquad \textit{if } j-i \geq 1, \\ \\ \delta_{kn}\nabla((i-j-n)\varpi_1 + \varpi_m)^* \\ \qquad \textit{if } j-i \leq -n, \\ \\ 0 \quad \textit{else}, \end{cases}$$

$$\mathrm{Ext}^k_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_m))(i), \mathcal{O}_{\mathcal{Q}}(j)) \simeq \begin{cases} \delta_{k0}\nabla((j-i-1)\varpi_1 + \varpi_m) \\ \qquad \textit{if } j-i \geq 1 \\ \\ \delta_{kn}\nabla((i-j-n)\varpi_1 + \varpi_{m+1})^* \\ \qquad \textit{if } j-i \leq -n \\ \\ 0 \quad \textit{else}. \end{cases}$$

(iii) Assume $n = 2m$ with m odd. Then

$$
\begin{aligned}
\operatorname{Ext}^k_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_{m+1})), \mathcal{L}(\nabla^P(\varpi_{m+1}))) &\simeq \delta_{k0}\Bbbk \\
&\simeq \operatorname{Ext}^k_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_m)), \mathcal{L}(\nabla^P(\varpi_m))), \\
\operatorname{Ext}^\bullet_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_{m+1})), \mathcal{L}(\nabla^P(\varpi_m))) &= 0 \\
&= \operatorname{Ext}^\bullet_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_m)), \mathcal{L}(\nabla^P(\varpi_{m+1}))), \\
\operatorname{Ext}^k_{\mathcal{Q}}(\mathcal{O}_{\mathcal{Q}}(i), \mathcal{O}_{\mathcal{Q}}(j)) &\simeq \begin{cases} \delta_{k0}\nabla((j-i)\varpi_1) \\ \quad \text{if } j-i \geq 0, \\ \\ \delta_{kn}\nabla((i-j-n)\varpi_1)^* \\ \qquad \simeq \delta_{kn}\Delta((i-j-n)\varpi_1) \\ \quad \text{if } j-i \leq -n, \end{cases} \\
\operatorname{Ext}^k_{\mathcal{Q}}(\mathcal{O}_{\mathcal{Q}}(i), \mathcal{L}(\nabla^P(\varpi_{m+1}))(j)) &\simeq \begin{cases} \delta_{k0}\nabla((j-i)\varpi_1 + \varpi_{m+1}) \\ \quad \text{if } j-i \geq 0, \\ \\ \delta_{kn}\nabla((i-j-1-n)\varpi_1 + \varpi_m)^* \\ \quad \text{if } j-i \leq -n-1 \\ \\ 0 \quad \text{else}, \end{cases} \\
\operatorname{Ext}^k_{\mathcal{Q}}(\mathcal{O}_{\mathcal{Q}}(i), \mathcal{L}(\nabla^P(\varpi_m))(j)) &\simeq \begin{cases} \delta_{k0}\nabla((j-i)\varpi_1 + \varpi_m) \\ \quad \text{if } j-i \geq 0 \\ \\ \delta_{kn}\nabla((i-j-1-n)\varpi_1 + \varpi_{m+1})^* \\ \quad \text{if } j-i \leq -n-1 \\ \\ 0 \quad \text{else}, \end{cases} \\
\operatorname{Ext}^k_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_{m+1}))(i), \mathcal{O}_{\mathcal{Q}}(j)) &\simeq \begin{cases} \delta_{k0}\nabla((j-i-1)\varpi_1 + \varpi_m) \\ \quad \text{if } j-i \geq 1 \\ \\ \delta_{kn}\nabla((i-j-n)\varpi_1 + \varpi_{m+1})^* \\ \quad \text{if } j-i \leq -n \\ \\ 0 \quad \text{else}, \end{cases}
\end{aligned}
$$

$$\mathrm{Ext}^k_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_m))(i), \mathcal{O}_{\mathcal{Q}}(j)) \simeq \begin{cases} \delta_{k0}\nabla((j-i-1)\varpi_1 + \varpi_{m+1}) \\ \quad \textit{if } j-i \geq 1 \\ \\ \delta_{kn}\nabla((i-j-n)\varpi_1 + \varpi_m)^* \\ \quad \textit{if } j-i \leq -n \\ \\ 0 \quad \textit{else.} \end{cases}$$

Proof: (i) One has

$$\begin{aligned} &\mathrm{Ext}^k_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_{m+1})), \mathcal{L}(\nabla^P(\varpi_{m+1}))) \\ &\quad \simeq \mathrm{H}^k(\mathcal{Q}, \mathcal{L}(\nabla^P(\varpi_{m+1}))^\vee \otimes_{\mathcal{Q}} \mathcal{L}(\nabla^P(\varpi_{m+1}))) \\ &\quad \simeq \mathrm{H}^k(\mathcal{Q}, \mathcal{L}((-\varpi_1) \otimes \nabla^P(\varpi_{m+1}) \otimes \nabla^P(\varpi_{m+1}))) \quad \text{by (4.1)} \end{aligned}$$

with $(-\varpi_1) \otimes \nabla^P(\varpi_{m+1}) \otimes \nabla^P(\varpi_{m+1}) \simeq \nabla^P(\frac{1}{2}(\varepsilon_2 + \cdots + \varepsilon_{m+1})) \otimes \nabla^P(\frac{1}{2}(\varepsilon_2 + \cdots + \varepsilon_{m+1}))$ admitting by (4.5) a filtration of P-modules of subquotients $\nabla^P(\varepsilon_2 + \cdots + \varepsilon_{m+1}) \simeq \nabla^P(-\varpi_1 + 2\varpi_{m+1})$, $\nabla^P(\varepsilon_2 + \cdots + \varepsilon_m) \simeq \nabla^P(-\varpi_1 + \varpi_m)$, $\ldots$, $\nabla^P(\varepsilon_2 + \varepsilon_3) \simeq \nabla^P(-\varpi_1 + \varpi_3)$, $\nabla^P(\varepsilon_2) \simeq \nabla^P(-\varpi_1 + \varpi_2)$, and $\nabla^P(\Bbbk) \simeq \Bbbk$, each appearing once. As

$$\begin{aligned} \mathrm{H}^\bullet(\mathcal{Q}, \mathcal{L}(\nabla^P(-\varpi_1 + 2\varpi_{m+1}))) &\simeq \mathrm{H}^\bullet(\mathcal{B}, \mathcal{L}(-\varpi_1 + 2\varpi_{m+1})) \\ &= 0 \\ &= \mathrm{H}^\bullet(\mathcal{B}, \mathcal{L}(-\varpi_1 + \varpi_m)) \\ &\simeq \mathrm{H}^\bullet(\mathcal{Q}, \mathcal{L}(\nabla^P(-\varpi_1 + \varpi_m))) \\ &\ldots \\ &= \mathrm{H}^\bullet(\mathcal{B}, \mathcal{L}(-\varpi_1 + \varpi_2)) \\ &\simeq \mathrm{H}^\bullet(\mathcal{Q}, \mathcal{L}(\nabla^P(-\varpi_1 + \varpi_2))), \end{aligned}$$

$$\mathrm{Ext}^k_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_{m+1})), \mathcal{L}(\nabla^P(\varpi_{m+1}))) \simeq \mathrm{H}^k(\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}) \simeq \delta_{k0}\Bbbk.$$

Next, from (4.4)

$$\begin{aligned}\mathrm{Ext}^k_{\mathcal{Q}}(\mathcal{O}_{\mathcal{Q}}(i),\mathcal{L}(\nabla^P(\varpi_{m+1}))(j)) &\simeq \mathrm{H}^k(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_{m+1}))(j-i))\\ &\simeq \begin{cases}\delta_{k0}\nabla((j-i)\varpi_1+\varpi_{m+1}) & \text{if } j-i\geq -1\\ \delta_{kn}\nabla((i-j-1-n)\varpi_1+\varpi_{m+1})^* & \text{if } j-i\leq -2\end{cases}\\ &\simeq \begin{cases}\delta_{k0}\nabla((j-i)\varpi_1+\varpi_{m+1}) & \text{if } j-i\geq 0\\ \delta_{kn}\nabla((i-j-1-n)\varpi_1+\varpi_{m+1})^* & \text{if } j-i\leq -n-1\\ 0 & \text{else.}\end{cases}\end{aligned}$$

Likewise,

$$\begin{aligned}\mathrm{Ext}^k_{\mathcal{Q}}(\mathcal{L}(\nabla^P(\varpi_{m+1}))(i),\mathcal{O}_{\mathcal{Q}}(j)) &\simeq \mathrm{H}^k(\mathcal{Q},\mathcal{L}(\nabla^P(\varpi_{m+1}))(j-i-1)) \text{ by (4.1)}\\ &\simeq \begin{cases}\delta_{k0}\nabla((j-i-1)\varpi_1+\varpi_{m+1}) & \text{if } j-i\geq 1\\ \delta_{kn}\nabla((i-j-n)\varpi_1+\varpi_{m+1})^* & \text{if } j-i\leq -n\\ 0 & \text{else.}\end{cases}\end{aligned}$$

(4.7) **Remark:** Recall from (1.7) the quotient $\pi:\hat{\nabla}_P(\Bbbk)\to \mathrm{hd}_{G_1P}\hat{\nabla}_P(\Bbbk) = L((w^P\bullet 0)^0)\otimes p\{(w^P)^{-1}\bullet(w^P\bullet 0)^1\}$. Regardless of characteristic $p>2$, one has $w^P\bullet 0 = -n\varpi_1$ by (3.4), and hence $(w^P)^{-1}\bullet(w^P\bullet 0)^1 = (r-n)\varpi_1$, where $r\in\mathbb{N}$ with $rp-n\in[0,p[$. It follows from [La] that $\mathcal{L}_{\mathcal{Q}}(\pi)$ always splits.

5° Kapranov's resolution

The present proof of the generation of the derived category $\mathrm{D}^b(\mathrm{coh}(\mathcal{Q}))$ by the $\mathcal{E}_w$, $w\in W^P$, depends on Kapranov's resolution of $\Delta_*\mathcal{O}_{\mathcal{Q}}$, $\Delta:\mathcal{Q}\to\mathcal{Q}\times_{\Bbbk}\mathcal{Q}$ the diagonal imbedding. We attempt to write down an outline of the construction, following Swan [Sw] and Böhning [Bö] as well as [Kap], to convince the reader that it can be done independent of characteristic $\neq 2$. Thus we continue to work with an arbitrary field $\Bbbk$ of characteristic $\neq 2$.

(5.1) If $n=2m+1$, put $\mathcal{S}=\mathcal{L}(\nabla^P(\varpi_{m+1}))^\vee\simeq\mathcal{L}(\nabla^P(\varpi_{m+1}))(-1)$. In case $n=2m$, put

$$\mathcal{S}_+=\mathcal{L}(\nabla^P(\varpi_{m+1}))^\vee\simeq\begin{cases}\mathcal{L}(\nabla^P(\varpi_{m+1}))(-1) & \text{if } m \text{ is even}\\ \mathcal{L}(\nabla^P(\varpi_m))(-1) & \text{if } m \text{ is odd}\end{cases}$$

and

$$\mathcal{S}_-=\mathcal{L}(\nabla^P(\varpi_{m}))^\vee\simeq\begin{cases}\mathcal{L}(\nabla^P(\varpi_{m}))(-1) & \text{if } m \text{ is even}\\ \mathcal{L}(\nabla^P(\varpi_{m+1}))(-1) & \text{if } m \text{ is odd.}\end{cases}$$

We will show

Theorem: . *(i) If n is odd, there is a resolution of $\Delta_*\mathcal{O}_\mathcal{Q}$*

$$\begin{aligned}0 \to \mathcal{S} \boxtimes \mathcal{S}(-n+1) \to \Psi_{n-1} \boxtimes \mathcal{O}_\mathcal{Q}(-n+1) \to \Psi_{n-2} \boxtimes \mathcal{O}_\mathcal{Q}(-n+2) \to \dots \\ \to \Psi_1 \boxtimes \mathcal{O}_\mathcal{Q}(-1) \to \mathcal{O}_{\mathcal{Q}\times_\Bbbk \mathcal{Q}} \to \Delta_*\mathcal{O}_\mathcal{Q} \to 0\end{aligned}$$

with some coherent Ψ_i, $i \in [1, n-1]$.

(ii) Assume $n = 2m$. If m is even, $\Delta_*\mathcal{O}_\mathcal{Q}$ admits a resolution

$$\begin{aligned}0 &\to (\mathcal{S}_+ \boxtimes \mathcal{S}_+(-n+1)) \oplus (\mathcal{S}_- \boxtimes \mathcal{S}_-(-n+1)) \\ &\to \Psi_{n-1} \boxtimes \mathcal{O}_\mathcal{Q}(-n+1) \to \Psi_{n-2} \boxtimes \mathcal{O}_\mathcal{Q}(-n+2) \to \dots \\ &\to \Psi_1 \boxtimes \mathcal{O}_\mathcal{Q}(-1) \to \mathcal{O}_{\mathcal{Q}\times_\Bbbk \mathcal{Q}} \to \Delta_*\mathcal{O}_\mathcal{Q} \to 0\end{aligned}$$

with some coherent Ψ_i, $i \in [1, n-1]$. If m is odd, $\Delta_*\mathcal{O}_\mathcal{Q}$ admits a resolution

$$\begin{aligned}0 &\to (\mathcal{S}_- \boxtimes \mathcal{S}_+(-n+1)) \oplus (\mathcal{S}_+ \boxtimes \mathcal{S}_-(-n+1)) \\ &\to \Psi_{n-1} \boxtimes \mathcal{O}_\mathcal{Q}(-n+1) \to \Psi_{n-2} \boxtimes \mathcal{O}_\mathcal{Q}(-n+2) \to \dots \\ &\to \Psi_1 \boxtimes \mathcal{O}_\mathcal{Q}(-1) \to \mathcal{O}_{\mathcal{Q}\times_\Bbbk \mathcal{Q}} \to \Delta_*\mathcal{O}_\mathcal{Q} \to 0\end{aligned}$$

with some coherent Ψ_i, $i \in [1, n-1]$.

(5.2) **Corollary:** *(i) If $n = 2m+1$, then $\mathcal{O}_\mathcal{Q}$, $\mathcal{O}_\mathcal{Q}(-1)$, ..., $\mathcal{O}_\mathcal{Q}(-n+1)$, $\mathcal{S}(-m)$ generate $\mathrm{D}^b(\mathrm{coh}\mathcal{Q})$.*

(ii) If $n = 2m$, then $\mathcal{O}_\mathcal{Q}$, $\mathcal{O}_\mathcal{Q}(-1)$, ..., $\mathcal{O}_\mathcal{Q}(-n+1)$, $\mathcal{S}_\pm(-m+1)$ generate $\mathrm{D}^b(\mathrm{coh}\mathcal{Q})$.

Proof: (i) By a general principle the resolution assures that $\mathcal{O}_\mathcal{Q}$, $\mathcal{O}_\mathcal{Q}(-1)$, ..., $\mathcal{O}_\mathcal{Q}(-n+1)$, $\mathcal{S}(-n+1) = \mathcal{L}(\nabla^P(\omega_{m+1}))(-n)$ generate $\mathrm{D}^b(\mathrm{coh}\mathcal{Q})$.

We have, however, from (4.2) a short exact sequence

$$0 \to \mathcal{L}(\nabla^P(\omega_{m+1}))(-1) \to \mathcal{O}_\mathcal{Q} \otimes \nabla(\omega_{m+1}) \to \mathcal{L}(\nabla^P(\omega_{m+1})) \to 0,$$

from which one obtains a short exact sequence $\forall k \in \mathbb{Z}$

$$0 \to \mathcal{L}(\nabla^P(\omega_{m+1}))(k-1) \to \mathcal{O}_\mathcal{Q}(k) \otimes \nabla(\omega_{m+1}) \to \mathcal{L}(\nabla^P(\omega_{m+1}))(k) \to 0,$$

and hence $\mathcal{L}(\nabla^P(\omega_{m+1}))(-n)$ belongs to the subcategory generated by $\mathcal{O}_\mathcal{Q}$, $\mathcal{O}_\mathcal{Q}(-1), \dots \mathcal{O}_\mathcal{Q}(-n+1)$ and $\mathcal{L}(\nabla^P(\omega_{m+1}))(-m-1) = \mathcal{S}(-m)$ by induction.

Likewise (ii).

(5.3) The construction of the resolution (5.1) requires Clifford algebras. We may assume by flat base change that $\Bbbk$ is algebraically closed. Let $C = C(E; q) = T_\Bbbk(E)/(v \otimes v - q(v) \mid v \in E)$ be the Clifford algebra of E over $\Bbbk$, which is naturally equipped with a structure of G-algebra. One has a G-linear decomposition $C = C_{\bar 0} \oplus C_{\bar 1}$ [Ch, II.1.0] such that

$$C_{\bar 0} \simeq \wedge^{\mathrm{even}} E = \coprod_{r\,\mathrm{even}} \wedge^r E \quad \text{and} \quad C_{\bar 1} \simeq \wedge^{\mathrm{odd}} E = \coprod_{r\,\mathrm{odd}} \wedge^r E.$$

If $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{so}(E) = \{x \in \mathfrak{gl}(E) \mid x^{\mathrm{t}}\mathbb{B} + \mathbb{B}x = 0\}$, there is a G-linear isomorphism $\eta : \wedge^2 E \to \mathfrak{g}$ via

$$v_1 \wedge v_2 \mapsto \mathbb{B}(v_2, ?)v_1 - \mathbb{B}(v_1, ?)v_2,$$

and an injective G-linear map $\xi : \wedge^2 E \to C_{\bar{0}}$ via

$$v_1 \wedge v_2 \mapsto v_1 v_2 - \frac{1}{2}\mathbb{B}(v_1, v_1).$$

It turns out that the composition $\xi \circ \eta^{-1} : \mathfrak{g} \to C_{\bar{0}}$ is an injective G-linear homomorphism of Lie algebras [FH, 20.1].

(5.4) Assume first $n = 2m + 1$. Recall from [Ch, II.2.6] that C has a central involution z, which is up to a scalar the product of the elements of an orthogonal basis of E. One has z fixed by G, and $C_{\bar{1}} = zC_{\bar{0}}$. Let $x_0 = \sqrt{-1}e_0$, $E^+ = \coprod_{i=1}^{m+1} \Bbbk e_i$, $E^- = \coprod_{i=1}^{m+1} \Bbbk e_{-i}$, $E' = E^+ \oplus E^-$, and put $C' = C(E')$ with respect to $q|_{E'}$. There is an isomorphism of $\Bbbk$-algebras $\psi : C' \to C_{\bar{0}}$ such that $\forall v_1, \dots, v_r \in E'$,

$$v_1 \dots v_r \mapsto \begin{cases} x_0 v_1 \dots v_r & \text{if } r \text{ is odd} \\ v_1 \dots v_r & \text{if } r \text{ is even.} \end{cases} \tag{1}$$

Consider $C'e'_-$ with $e'_- = e_{-1}e_{-2} \dots e_{-m-1}$. As $q(e_{-i}) = 0\ \forall i \in [1, m+1]$, $e_{-i}e'_- = 0$, and hence there is a $\Bbbk$-linear isomorphism $C'e'_- \simeq \wedge E^+$ via $v_1 \dots v_r e'_- \leftarrow\!\shortmid v_1 \wedge \dots \wedge v_r$. Recall also from [Ch, II.2.1 and 2.2] that $C'e'_-$ is a minimal left ideal of C' and that C' is the matrix algebra on $C'e'_-$ under the left multiplication: $C' \simeq \mathbf{Mod}_{\Bbbk}(C'e'_-, C'e'_-)$. Thus, transferring through ψ, let

$$e_- = \begin{cases} \sqrt{-1}e_0 e_{-m-1} \dots e_{-1} & \text{if } m \text{ is even} \\ e_{-m-1} \dots e_{-1} & \text{if } m \text{ is odd,} \end{cases}$$

and put $\rho_+ = C_{\bar{0}}e_-$. Then ρ_+ affords a simple module for $C_{\bar{0}}$, and hence there are $\Bbbk$-linear isomorphisms

$$C_{\bar{1}} \simeq C_{\bar{0}} \simeq \mathbf{Mod}_{\Bbbk}(\rho_+, \rho_+) \simeq \rho_+ \otimes \rho_+^* \tag{2}$$

such that, if $(y_1, \dots, y_{2^{m+1}})$ is a $\Bbbk$-linear basis of ρ_+ with $(y_1^*, \dots, y_{2^{m+1}}^*)$ its dual basis,

$$zx \leftarrow\!\shortmid x \mapsto l_x \mathrm{id}_{\rho_+} \mapsto \sum_{r=1}^{2^{m+1}} l_x y_r \otimes y_r^*,$$

where l_x is the multiplication by x from the left.

Assume next that $n = 2m$. As above, Ce_- with $e_- = \prod_{i=1}^{m+1} e_{-i}$ forms a simple left ideal of C such that

$$C \simeq \mathbf{Mod}_{\Bbbk}(Ce_-, Ce_-) \tag{3}$$

under the left multiplication. Put $E^+ = \coprod_{i=1}^{m+1} \Bbbk e_i$, $E^- = \coprod_{i=1}^{m+1} \Bbbk e_{-i}$, and let $C^\pm = C(E^\pm)$ with respect to $q|_{E^\pm}$. As $e_{-i}e_- = 0\ \forall i \in [1, m+1]$, there are $\Bbbk$-linear isomorphisms

$$Ce_- \simeq C^+e_- = (C^+_{\bar 0} \oplus C^+_{\bar 1})e_- = C^+_{\bar 0}e_- \oplus C^+_{\bar 1}e_- \simeq C^+_{\bar 0} \oplus C^+_{\bar 1}$$

with $C_{\bar 0}$ stabilizing under the left multiplication both $C^+_{\bar 0}e_-$ and $C^+_{\bar 1}e_-$ while $C_{\bar 1}$ exchanging those. If we put $\rho_+ = C^+_{\bar 0}e_-$ and $\rho_- = C^+_{\bar 1}e_-$, one has under the identification (3)

$$\begin{aligned} C_{\bar 0} &= \mathbf{Mod}_\Bbbk(\rho_+, \rho_+) \oplus \mathbf{Mod}_\Bbbk(\rho_-, \rho_-) \simeq (\rho_+^* \otimes \rho_+) \oplus (\rho_-^* \otimes \rho_-), \\ C_{\bar 1} &= \mathbf{Mod}_\Bbbk(\rho_+, \rho_-) \oplus \mathbf{Mod}_\Bbbk(\rho_-, \rho_+) \simeq (\rho_+^* \otimes \rho_-) \oplus (\rho_-^* \otimes \rho_+). \end{aligned}$$

One checks

Lemma: . *(i) Assume $n = 2m+1$. $\forall k \in [-m-1, m+1]$, let r_{e_k} be the right multiplication by e_k on C, and let $l^*_{e_k} f = f(e_k?), \mathrm{r}^*_{e_k} f = f(?e_k)$, $f \in C^*$. There is a commutative diagram*

$$\begin{tikzcd}
C_{\bar 0} \arrow[r, "\sim"] \arrow[d, "l_{e_k}"'] & \rho_+ \otimes \rho_+^* \arrow[dd, shift right, "l_{ze_k} \otimes \rho_+^*"'] \arrow[dd, shift left, "\rho_+ \otimes \ell^*_{ze_k}"] & C_{\bar 0} \arrow[l, "\sim"'] \arrow[d, "\mathrm{r}_{e_k}"] \\
C_{\bar 1} \arrow[d, "z"', "\sim"] & & C_{\bar 1} \arrow[d, "\sim"', "z"] \\
C_{\bar 0} \arrow[r, "\sim"] \arrow[d, "z"', "\sim"] & \rho_+ \otimes \rho_+^* \arrow[dd, shift right, "l_{ze_k} \otimes \rho_+^*"'] \arrow[dd, shift left, "\rho_+ \otimes l^*_{ze_k}"] & C_{\bar 0} \arrow[l, "\sim"'] \arrow[d, "\sim"', "z"] \\
C_{\bar 1} \arrow[d, "l_{e_k}"'] & & C_{\bar 1} \arrow[d, "\mathrm{r}_{e_k}"] \\
C_{\bar 0} \arrow[r, "\sim"'] & \rho_+ \otimes \rho_+^* & C_{\bar 0}. \arrow[l, "\sim"]
\end{tikzcd}$$

If we write $\bar i = \begin{cases} \bar 1 & \textit{if } i \textit{ is odd} \\ \bar 0 & \textit{if } i \textit{ is even} \end{cases}$ $\forall i \in \mathbb{N}$, *dualizing the above yields a commutative diagram*

$$\begin{tikzcd}
C^*_{\overline{i+1}} \arrow[r, "\sim"] \arrow[d, "l^*_{e_k}"'] & \rho_+^* \otimes \rho_+ \arrow[d, shift right, "l^*_{ze_k} \otimes \rho_+"'] \arrow[d, shift left, "\rho_+^* \otimes l_{ze_k}"] & C^*_{\overline{i+1}} \arrow[l, "\sim"'] \arrow[d, "\mathrm{r}^*_{e_k}"] \\
C^*_{\bar i} \arrow[r, "\sim"'] & \rho_+^* \otimes \rho_+ & C^*_{\bar i}. \arrow[l, "\sim"]
\end{tikzcd}$$

(ii) Assume $n = 2m$. $\forall k \in [-m-1, m+1] \setminus 0$, there is a commutative diagram

$$\begin{tikzcd}
C_{\bar{0}} \arrow[r, "\sim"] \arrow[d, "l_{e_k}"'] & (\rho_+^* \otimes \rho_+) \oplus (\rho_-^* \otimes \rho_-) \arrow[d, "(\rho_+^* \otimes l_{e_k}) \oplus (\rho_-^* \otimes l_{e_k})"] \\
C_{\bar{1}} \arrow[r, "\sim"] \arrow[d, "l_{e_k}"'] & (\rho_+^* \otimes \rho_-) \oplus (\rho_-^* \otimes \rho_+) \arrow[d, "(\rho_+^* \otimes l_{e_k}) \oplus (\rho_-^* \otimes l_{e_k})"] \\
C_{\bar{0}} \arrow[r, "\sim"] \arrow[d, "\mathrm{r}_{e_k}"'] & (\rho_+^* \otimes \rho_+) \oplus (\rho_-^* \otimes \rho_-) \arrow[d, "(l_{e_k}^* \otimes \rho_+) \oplus (l_{e_k}^* \otimes \rho_-)"] \\
C_{\bar{1}} \arrow[r, "\sim"] \arrow[d, "\mathrm{r}_{e_k}"'] & (\rho_-^* \otimes \rho_+) \oplus (\rho_+^* \otimes \rho_-) \arrow[d, "(l_{e_k}^* \otimes \rho_+) \oplus (l_{e_k}^* \otimes \rho_-)"] \\
C_{\bar{0}} \arrow[r, "\sim"] & (\rho_+^* \otimes \rho_+) \oplus (\rho_-^* \otimes \rho_-).
\end{tikzcd}$$

(5.5) **Proposition:** *(i) If $n = 2m+1$, the structure of $\mathfrak{g}$-module on ρ_+ extends to a structure of G-module such that $\rho_+ \simeq \nabla(\varpi_{m+1})$.*

(ii) If $n = 2m$, the structure of $\mathfrak{g}$-module on $\rho_\pm$ extends to a structure of G-module such that

$$\rho_+ \simeq \begin{cases} \nabla(\varpi_{m+1}) & \text{if } m \text{ is odd} \\ \nabla(\varpi_m) & \text{if } m \text{ is even,} \end{cases}$$

$$\rho_- \simeq \begin{cases} \nabla(\varpi_m) & \text{if } m \text{ is odd} \\ \nabla(\varpi_{m+1}) & \text{if } m \text{ is even.} \end{cases}$$

Proof: (i) Put $e_+ = e_1 \dots e_{m+1}$. One finds the weight of $e_+e_- \in \rho_+$ with respect to $\mathrm{Lie}(T)$ is ω_{m+1}, and that e_+e_- is annihilated by $\mathrm{Lie}(U^+)$, U^+ the unipotent radical of the Borel subgroup opposite to B. There is then a nonzero homomorphism of $\mathrm{Dist}(G_1)$-modules

$$\bar{\Delta}(\omega_{m+1}) = \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(B_1^+)} \omega_{m+1} \to \rho_+.$$

On the other hand, as ω_{m+1} is minuscule, $\nabla(\omega_{m+1})$ is irreducible over G_1. Then $\nabla(\omega_{m+1}) = \mathrm{hd}_{G_1}(\bar{\Delta}(\omega_{m+1}))$, and hence $\nabla(\omega_{m+1}) \simeq \rho_+$ by dimension (4.1).

Likewise (ii).

(5.6) Let (e_k^*) be a basis for E^* dual to (e_k).

Corollary: . *(i) If $n = 2m+1$, the morphism*

$$\sum_{k=-m-1}^{m+1} l_{ze_k} \otimes e_k^* : \rho_+ \otimes \mathcal{O}_{\mathcal{Q}}(-1) \to \rho_+ \otimes \mathcal{O}_{\mathcal{Q}}$$

is G-equivariant, inducing an exact sequence of G-equivariant sheaves

$$\cdots \to \rho_+ \otimes \mathcal{O}_{\mathcal{Q}}(-2) \xrightarrow{\sum_k l_{ze_k} \otimes e_k^*} \rho_+ \otimes \mathcal{O}_{\mathcal{Q}}(-1) \xrightarrow{\sum_k l_{ze_k} \otimes e_k^*} \rho_+ \otimes \mathcal{O}_{\mathcal{Q}} \to \mathcal{L}(\nabla^P(\omega_{m+1})) \to 0.$$

In particular,

$$\mathrm{coker}(\sum_{k=-m-1}^{m+1} l_{ze_k} \otimes e_k^* : \rho_+ \otimes \mathcal{O}_{\mathcal{Q}}(-1) \to \rho_+ \otimes \mathcal{O}_{\mathcal{Q}}) \simeq \mathcal{L}(\nabla^P(\omega_{m+1})).$$

Dualizing, one has also an exact sequence

$$\cdots \to \rho_+^* \otimes \mathcal{O}_{\mathcal{Q}}(-2) \to \rho_+^* \otimes \mathcal{O}_{\mathcal{Q}}(-1) \to \rho_+^* \otimes \mathcal{O}_{\mathcal{Q}} \to \mathcal{L}(\nabla^P(\omega_{m+1})) \to 0.$$

(ii) If $n = 2m$, the morphisms

$$\sum_{k\in[-2m,2m]\setminus 0} e_k \otimes e_k^* : \rho_- \otimes \mathcal{O}_{\mathcal{Q}}(-1) \to \rho_+ \otimes \mathcal{O}_{\mathcal{Q}}$$

and

$$\sum_{k\in[-2m,2m]\setminus 0} e_k \otimes e_k^* : \rho_+ \otimes \mathcal{O}_{\mathcal{Q}}(-1) \to \rho_- \otimes \mathcal{O}_{\mathcal{Q}}$$

are both G-equivariant, and hence

$$\mathrm{coker}(\sum_{k\in[-2m,2m]\setminus 0} e_k \otimes e_k^* : \rho_- \otimes \mathcal{O}_{\mathcal{Q}}(-1) \to \rho_+ \otimes \mathcal{O}_{\mathcal{Q}}) \simeq \begin{cases} \mathcal{L}(\nabla^P(\varpi_m)) \\ \text{if } m \text{ is even} \\ \\ \mathcal{L}(\nabla^P(\varpi_{m+1})) \\ \text{if } m \text{ is odd,} \end{cases}$$

$$\mathrm{coker}(\sum_{k\in[-2m,2m]\setminus 0} e_k \otimes e_k^* : \rho_- \otimes \mathcal{O}_{\mathcal{Q}}(-1) \to \rho_+ \otimes \mathcal{O}_{\mathcal{Q}}) \simeq \begin{cases} \mathcal{L}(\nabla^P(\varpi_{m+1})) \\ \text{if } m \text{ is even} \\ \\ \mathcal{L}(\nabla^P(\varpi_m)) \\ \text{if } m \text{ is odd.} \end{cases}$$

Dualizing, one has also

$$\operatorname{coker}(\sum_k (l^*_{e_k} \otimes e^*_k) : \rho^*_+ \otimes \mathcal{O}_{\mathcal{Q}}(-n-1) \to \rho^*_- \otimes \mathcal{O}_{\mathcal{Q}}(-n)) \simeq \mathcal{L}(\nabla^P(\varpi_m) \otimes (-n\varpi_1)),$$

$$\operatorname{coker}(\sum_k (l^*_{e_k} \otimes e^*_k) : \rho^*_- \otimes \mathcal{O}_{\mathcal{Q}}(-n-1) \to \rho^*_+ \otimes \mathcal{O}_{\mathcal{Q}}(-n)) \simeq \mathcal{L}(\nabla^P(\varpi_{m+1}) \otimes (-n\varpi_1)).$$

Proof: (i) Upon checking the G-equivariance of

$$\sum_{k=-m-1}^{m+1} l_{ze_k} \otimes e^*_k : \rho_+ \otimes \mathcal{O}_{\mathcal{Q}}(-1) \to \rho_+ \otimes \mathcal{O}_{\mathcal{Q}}$$

one sees that it is by (5.5.i) induced from a P-linear morphism

$$(\sum_k l_{ze_k} \otimes e^*_k)(e) : \nabla(\omega_{m+1}) \otimes (-\omega_1) \to \nabla(\omega_{m+1}),$$

e being the point eP on $\mathcal{Q} = G/P$. By the unicity of such it must coincide up to a nonzero scalar with the one obtained in (4.1), and hence yields an exact sequence

$$\cdots \to \nabla(\omega_{m+1}) \otimes (-2\omega_1) \xrightarrow{(\sum_k l_{ze_k} \otimes e^*_k)(e) \otimes (-\omega_1)} \nabla(\omega_{m+1}) \otimes (-\omega_1) \xrightarrow{(\sum_k l_{ze_k} \otimes e^*_k)(e)} \nabla(\omega_{m+1}) \to \nabla^P(\omega_{m+1}) \to 0,$$

sheafification of which reads the exact sequence

$$\cdots \to \rho_+ \otimes \mathcal{O}_{\mathcal{Q}}(-2) \xrightarrow{\sum_k l_{ze_k} \otimes e^*_k} \rho_+ \otimes \mathcal{O}_{\mathcal{Q}}(-1) \xrightarrow{\sum_k \psi^{-1}(ze_k) \otimes e^*_k} \rho_+ \otimes \mathcal{O}_{\mathcal{Q}} \to \mathcal{L}(\nabla^P(\omega_{m+1})) \to 0.$$

Dualizing, $\sum_k l^*_{ze_k} \otimes e^*_k : \rho^*_+ \otimes \mathcal{O}_{\mathcal{Q}} \to \rho^*_+ \otimes \mathcal{O}_{\mathcal{Q}}(1)$ is induced by a P-linear map $\nabla(\omega_{m+1}) \to \nabla(\omega_{m+1}) \otimes \omega_1$ as $\rho^*_+ \simeq \nabla(\omega_{m+1})^* \simeq \nabla(\omega_{m+1})$, and hence yields the desired exact sequence.

Likewise (ii).

(5.7) Let next $A = \mathrm{T}_{\Bbbk}(E \oplus \Bbbk\hbar)/(v \otimes v - q(v)\hbar, v \otimes \hbar - \hbar \otimes v \mid v \in E)$ with $\hbar$ an indeterminate. For the Clifford algebra $C = C(E)$ there is an $\Bbbk$-linear isomorphism

$$A \simeq C \otimes \Bbbk[\hbar]. \tag{1}$$

We make A into a graded $\Bbbk$-algebra by giving each element of E (resp. $\hbar$) degree 1 (resp. 2): $\forall i \in \mathbb{N}$, there is a $\Bbbk$-linear isomorphism

$$A_i \simeq \coprod_{j+2k=i} (\wedge^j E) \otimes \Bbbk\hbar^k. \tag{2}$$

$\forall j \in \mathbb{N}$, consider $d_j = \sum_k \mathrm{r}_{e_k} \otimes e_k^* : A_j \otimes \mathcal{O}_{\mathcal{Q}}(j) \to A_{j+1} \otimes \mathcal{O}_{\mathcal{Q}}(j+1)$ with r_{e_k} being the multiplication to the right on A_j by e_k. Then $d_{j+1} \circ d_j = 0$. Its dual complex $\hat{A}^*$:

$$\cdots \to A_{j+1}^* \otimes \mathcal{O}_{\mathcal{Q}}(-j-1) \xrightarrow{\sum_k \mathrm{r}_{e_k}^* \otimes e_k^*} A_j^* \otimes \mathcal{O}_{\mathcal{Q}}(-j) \xrightarrow{\sum_k \mathrm{r}_{e_k}^* \otimes e_k^*} \cdots$$
$$\xrightarrow{\sum_k \mathrm{r}_{e_k}^* \otimes e_k^*} A_1^* \otimes \mathcal{O}_{\mathcal{Q}}(-1) \xrightarrow{\sum_k \mathrm{r}_{e_k}^* \otimes e_k^*} A_0^* \otimes \mathcal{O}_{\mathcal{Q}} \to 0$$

is exact [Sw, Lem. 7.3], called the Tate resolution of $A_0^* \otimes \mathcal{O}_{\mathcal{Q}} \simeq \mathcal{O}_{\mathcal{Q}}$.

(5.8) Now let $\Psi_i = \ker(\sum_k \mathrm{r}_{e_k}^* \otimes e_k^* : A_i^* \otimes \mathcal{O}_{\mathcal{Q}} \to A_{i-1}^* \otimes \mathcal{O}_{\mathcal{Q}}(1))$ $\forall i \in \mathbb{N}$. Using the double complex

$$\begin{tikzcd}
 & 0 & & & \\
\cdots \arrow[r] & \mathcal{O}_{\mathcal{Q}}(2) \boxtimes \mathcal{O}_{\mathcal{Q}}(-2) \arrow[u] \arrow[rr] & & 0 & \\
\cdots \arrow[r] & A_1^* \otimes \mathcal{O}_{\mathcal{Q}}(1) \boxtimes \mathcal{O}_{\mathcal{Q}}(-2) \arrow[u, "(\sum_k \mathrm{r}_{e_k}^* \otimes e_k^*) \boxtimes \mathcal{O}_{\mathcal{Q}}(-2)"] \arrow[rr, "\sum_k l_{e_k}^* \otimes \mathcal{O}_{\mathcal{Q}}(1) \boxtimes e_k^*"] & & \mathcal{O}_{\mathcal{Q}}(1) \boxtimes \mathcal{O}_{\mathcal{Q}}(-1) \arrow[u] \arrow[r] & 0 \\
\cdots \arrow[r] & A_2^* \otimes \mathcal{O}_{\mathcal{Q}} \boxtimes \mathcal{O}_{\mathcal{Q}}(-2) \arrow[u, "(\sum_k \mathrm{r}_{e_k}^* \otimes e_k^*) \boxtimes \mathcal{O}_{\mathcal{Q}}(-2)"] \arrow[rr, "\sum_k l_{e_k}^* \otimes \mathcal{O}_{\mathcal{Q}} \boxtimes e_k^*"'] & & A_1^* \otimes \mathcal{O}_{\mathcal{Q}} \boxtimes \mathcal{O}_{\mathcal{Q}}(-1) \arrow[u, "(\sum_k \mathrm{r}_{e_k}^* \otimes e_k^*) \boxtimes \mathcal{O}_{\mathcal{Q}}(-1)"] \arrow[r, "\sum_k l_{e_k}^* \otimes \mathcal{O}_{\mathcal{Q}} \boxtimes e_k^*"'] & \mathcal{O}_{\mathcal{Q} \times \mathcal{Q}} \arrow[u] \arrow[r] & 0,
\end{tikzcd}$$

one shows that the complex $\hat{\Psi}^\bullet$:

$$\cdots \to \Psi_i \boxtimes \mathcal{O}_{\mathcal{Q}}(-i) \to \Psi_{i-1} \boxtimes \mathcal{O}_{\mathcal{Q}}(-i+1) \to \cdots \to \Psi_0 \boxtimes \mathcal{O}_{\mathcal{Q}}$$

gives a resolution of $\Delta_* \mathcal{O}_{\mathcal{Q}}$.

(5.9) By definition the A^*_j stabilize after n: $\forall j \geq n+1$, one has $\mathbb{k}$-linear isomorphisms

$$
\begin{aligned}
A^*_j &\simeq \coprod_{r+2s=j} \{\wedge^r(E^*) \otimes \mathbb{k}\hbar^{(s)}\} \quad \text{with } (\hbar^{(s)}) \text{ the graded dual basis of } (\hbar^s) \qquad (1)\\
&\simeq \begin{cases} \coprod_{r \text{ even}} \wedge^r(E^*) & \text{if } j \text{ is even} \\ \coprod_{r \text{ odd}} \wedge^r(E^*) & \text{if } j \text{ is odd} \end{cases}\\
&\simeq C^*_{\bar{j}} \quad \text{with } \bar{j} \equiv j \mod 2\\
&\simeq \begin{cases} \rho_+ \otimes \rho^*_+ & \text{if } n = 2m+1 \\ (\rho^*_+ \otimes \rho_+) \oplus (\rho^*_- \otimes \rho_-) & \text{if } n = 2m \text{ and } j \text{ even} \\ (\rho^*_+ \otimes \rho_-) \oplus (\rho^*_- \otimes \rho_+) & \text{if } n = 2m \text{ and } j \text{ odd.} \end{cases}
\end{aligned}
$$

Put $\mathcal{K} = \ker(\Psi_{n-1} \boxtimes \mathcal{O}_{\mathcal{Q}}(-n+1) \to \Psi_{n-2} \boxtimes \mathcal{O}_{\mathcal{Q}}(-n+2))$. By the exactness of $\hat{\Psi}^\bullet$

$$
\begin{aligned}
\mathcal{K} &\simeq \operatorname{im}(\Psi_n \boxtimes \mathcal{O}_{\mathcal{Q}}(-n) \to \Psi_{n-1} \boxtimes \mathcal{O}_{\mathcal{Q}}(-n+1)) \qquad (2)\\
&\simeq \operatorname{coker}(\Psi_{n+1} \boxtimes \mathcal{O}_{\mathcal{Q}}(-n-1) \to \Psi_n \boxtimes \mathcal{O}_{\mathcal{Q}}(-n))
\end{aligned}
$$

while the exactness of the Tate-Swan resolution implies

$$
\begin{aligned}
\Psi_i &= \ker(A^*_i \otimes \mathcal{O}_{\mathcal{Q}} \to A^*_{i-1} \otimes \mathcal{O}_{\mathcal{Q}}(1)) \simeq \operatorname{im}(A^*_{i+1} \otimes \mathcal{O}_{\mathcal{Q}}(-1) \to A^*_i \otimes \mathcal{O}_{\mathcal{Q}}) \qquad (3)\\
&\simeq \operatorname{coker}(A^*_{i+2} \otimes \mathcal{O}_{\mathcal{Q}}(-2) \to A^*_{i+1} \otimes \mathcal{O}_{\mathcal{Q}}(-1)) \quad \forall i \in \mathbb{N}.
\end{aligned}
$$

Consider the double complex $\mathcal{K}^{\bullet\bullet}$

$$
\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] \\
\cdots \arrow[r, dashed] & (A^*_{n+3} \otimes \mathcal{O}_{\mathcal{Q}}(-2)) \boxtimes \mathcal{O}_{\mathcal{Q}}(-n-1) \arrow[rr, "(\sum_k \ell^*_{e_k} \otimes \mathcal{O}_{\mathcal{Q}}(-2)) \boxtimes e^*_k"] \arrow[d, "\sum_k (r^*_{e_k} \otimes e^*_k) \boxtimes \mathcal{O}_{\mathcal{Q}}(-n-1)"'] & & (A^*_{n+2} \otimes \mathcal{O}_{\mathcal{Q}}(-2)) \boxtimes \mathcal{O}_{\mathcal{Q}}(-n) \arrow[d, "\sum_k (r^*_{e_k} \otimes e^*_k) \boxtimes \mathcal{O}_{\mathcal{Q}}(-n)"'] \\
\cdots \arrow[r, dashed] & (A^*_{n+2} \otimes \mathcal{O}_{\mathcal{Q}}(-1)) \boxtimes \mathcal{O}_{\mathcal{Q}}(-n-1) \arrow[rr, "(\sum_k \ell^*_{e_k} \otimes \mathcal{O}_{\mathcal{Q}}(-1)) \boxtimes e^*_k"'] & & (A^*_{n+1} \otimes \mathcal{O}_{\mathcal{Q}}(-1)) \boxtimes \mathcal{O}_{\mathcal{Q}}(-n).
\end{tikzcd}
$$

By (3) the total complex $\mathrm{Tot}(\mathcal{K}^{\bullet\bullet})$ is quasi-isomorphic to $\mathcal{K}$. On the other hand, if $n = 2m+1$, $\mathcal{K}^{\bullet\bullet}$ reads by (5.4.i)

$$\begin{tikzcd}
 & \vdots \arrow[dr] & & \\
\vdots \arrow[dr, "\sum_k (\ell_{ze_k} \otimes e_k^*) \boxtimes (\rho_+^* \otimes \mathcal{O}_{\mathcal{Q}}(-n-1))"] & & (\rho_+ \otimes \mathcal{O}_{\mathcal{Q}}(-1)) \boxtimes (\rho_+^* \otimes \mathcal{O}_{\mathcal{Q}}(-n-1)) \arrow[dr, "(\rho_+ \otimes \mathcal{O}_{\mathcal{Q}}(-1)) \boxtimes \sum_k (\psi^{-1}(ze_k)^* \otimes e_k^*)"] & \\
 & (\rho_+ \otimes \mathcal{O}_{\mathcal{Q}}(-2)) \boxtimes (\rho_+^* \otimes \mathcal{O}_{\mathcal{Q}}(-n-1)) \arrow[ur] \arrow[dr] & & (\rho_+ \otimes \mathcal{O}_{\mathcal{Q}}(-1)) \boxtimes (\rho_+^* \otimes \mathcal{O}_{\mathcal{Q}}(-n)), \\
\cdots \arrow[ur, "(\rho_+ \otimes \mathcal{O}_{\mathcal{Q}}(-2)) \boxtimes \sum_k (\psi^{-1}(ze_k)^* \otimes e_k^*)"'] & & (\rho_+ \otimes \mathcal{O}_{\mathcal{Q}}(-2)) \boxtimes (\rho_+^* \otimes \mathcal{O}_{\mathcal{Q}}(-n)) \arrow[ur, "\sum_k (\ell_{ze_k} \otimes e_k^*) \boxtimes (\rho_+^* \otimes \mathcal{O}_{\mathcal{Q}}(-n))"] & \\
 & \cdots \arrow[ur] & &
\end{tikzcd}$$

and hence $\mathrm{Tot}(\mathcal{K}^{\bullet\bullet})$ is by (5.6.i) quasi-isomorphic to $\mathcal{L}(\nabla^P(\varpi_{m+1})) \boxtimes \mathcal{L}(\nabla^P(\varpi_{m+1}))(-n+1) = \mathcal{S} \boxtimes \mathcal{S}(-n+1)$, establishing (5.1.i). Likewise if n is even.

(5.10) Back to the general set-up, Samokhin [Sa, Lem.13] shows for $p > h$ that if the distinct indecomposable direct summands of $F_*\mathcal{O}_{G/P}$ form a semi-orthogonal sequence, $F_*\mathcal{O}_{G/P}$ Karoubian generates $\mathrm{D}^b(\mathrm{coh}(G/P))$. For quadrics our $\mathcal{E}_w$'s, $w \in W^P$, indeed constitute for $p \geq h$ the distinct indecomposable direct summands of $F_*\mathcal{O}_Q$ by [La].

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